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Investigatio perturbationum quibus planetarum motus ob actionem eorum mutuam afficiuntur.

Leonhard Euler

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INVESTIGATIO

PERTURBATIONUM

Quibus Planetarum motus ob actionem eorum mutuam afficiuntur.

Autore LEONARDO EULERO, Matheseos Professore, Academiarum Parisiensis, Berolinensis & Petropolitanæ Socio.

Sidera quod tantis cieant se viribus æquis
In motu terræ plurima signa docent.

Hæc Dissertatio meruit Præmium duplicatum anno M.DCC.LVI.

PRÆFATIO.

PLANETAS non ſolum ad Solem ſecundum inverſam diſtantiarum rationem duplicatam impelli, ſed etiam ſimili ratione ſe mutuo incitare ex perturbationibus motuum Saturni & Jovis manifeſto eſt perſpectum. Cum enim Tabulæ Aſtronomicæ ita conſtrui ſoleant, quaſi planetæ ad ſolum Solem ſollicitati ſecundum regulas Keplerianas in ellipſibus revolverentur, ſi ex iis loca Saturni vel etiam Jovis definiantur ea nonnunquam ad plura minuta prima à veritate aberrare deprehenduntur; neque jam ullum eſt dubium, quin iſti errores ab actione mutua, qua hi duo planetæ ſe invicem impellunt, proficiſcantur. Reliquorum quidem planetarum motus ac præcipue terræ regulis illis Keplerianis magis eſt conformis, ac ſi quando errores in eorum motu à Tabulis occurrunt, incertum plerumque eſt, utrum illi vel non recte conſtitutis Tabularum elementis, vel ipſarum obſervationum imperfectioni cuipiam potius ſint tribuendi, quam ipſi Theoriæ, cui Tabulæ innituntur. Interim tamen jam ipſa tabularum ratio, qua pro quolibet planeta tam lineæ abſidum quam lineæ nodorum motus peculiaris aſſignatur, manifeſtum indicium aberrationis cujuſdam à Theoria continet: ſi enim planetæ nullam aliam impulſionem præter eam qua ſecundum rationem quadrati diſtantiarum inverſam ad ſolem urgentur, ſuſtinerent, non ſolum circa ſolem tanquam focum perfectas deſcriberent ellipſes, ſed etiam perpetuo in eodem plano ferrentur, axeſque iſtarum ellipſium omnino fixi manerent; neque idcirco linea abſidum neque linea nodorum ulli obnoxia foret mutationi, ſaltem reſpectu ſtellarum fixarum. Ad hanc quoque normam computatæ ſunt à ſtretio Tabulæ Carolinæ, in quibus tam apheliis quam nodis ſingulorum planetarum in cœlo ſidereo loca fixa aſſignantur; at vero inſignis harum tabularum à veritate diſſenſus mox luculenter monſtravit, huic hypotheſi locum concedi non poſſe. Cum igitur certum ſit tam aphelium quam linea nodorum cujuſque planetæ motu peculiari per cœlum proferri, atque etiam reſpectu ſtellarum fixarum continuo mutari; minime amplius dubitare licet, quin præter eam vim conſtantem, qua ſinguli planetæ ad ſolem pelluntur, aliæ quoque vires in eos effectum quempiam exerant. Quemadmodum enim in Saturno & Jove præter alias perturbationes ab eorum ac-

tione mutua utriuſque lineæ abſidum & nodorum certus imprimi motus eſt inventus obſervationibus ſatis conſentaneus, ita multo minus dubitare poterimus, quin ſimilis variatio in apheliis & nodis reliquorum quoque planetarum ab eorum actione mutua proficiſcatur, etiamſi in cæteris motus horum planetarum elementis nulla alteratio perciperetur. Sunt autem effectus talium virium in loca apheliorum & nodorum ita comparati, ut etiamſi ſint minimi, tamen cum tempore continuo creſcant, & poſt ſatis longum intervallum ſenſibiles evadant dum reliquæ perturbationes inde oriundæ ſunt periodicæ, & poſt certas revolutiones iterum penitus in nihilum redigantur, unde fit ut ſi ſint minimæ, percipi omnino non poſſint. Interim tamen Theoria motus telluris, cujus elementa per obſervationes Solis multo accuratius definire licet quam reliquorum planetarum, haud obſcura talium minimarum perturbationum ſigna exhibet, dum excentricitas ejus orbitæ prouti alia atque alia tempeſtate per obſervationes inveſtigatur, modo aliquantum major modo minor deprehenditur, quæ inconſtantia ad integrum minutum aſſurgere videtur. Quibus perpenſis palam omnino eſt non ſolum motum Saturni ac Jovis, ſed etiam reliquorum planetarum ab aliis viribus præter eam qua lege conſtanti ad Solem pelluntur, perturbari, earumque adeo effectum ab Astronomis manifesto esse observatum. Quamvis enim Astronomorum Princeps Halleyus Mercurii motum ab hujusmodi perturbationibus prorsus immunem sit arbitratus, propterea quod ob summam solis vicinitatem reliquorum planetarum vires præ vi Solis quasi evanescere crediderit, qua sententia fretus in tabulis suis etiam neque Aphelio Mercurii neque ejus lineæ nodorum motum ullum respectu stellarum fixarum adscripsit: tamen ex postremo potissimum transitu hujus planetæ per solem Astronomi didicerunt Halleyi Tabulas insigni emendatione ex hac parte indigere, dum aphelio Mercurii motum annuum quasi 55″, ejusque nodo 45″ ratione æquinoxii tribui debere est compertum; ex quo manifestum est etiam Mercurium actiones reliquorum planetarum sentire ob easque certis perturbationibus esse obnoxium.

Cum igitur extra dubium sit positum planetas in se invicem attrahendo agere, dispiciendum est quamnam rationem eorum vires respectu ad distantias habito sequantur. Ac præterquam quod constantia naturæ eandem legem quadratis distantiarum reciproce proportionalem, quam in Sole stabilitam cernimus, exigere videtur, Theoria Lunæ atque Satellitum Jovis & Saturni hanc suspicionem plenissime

confirmat, ita ut amplius dubitare non liceat, quin Jupiter Saturnusque suos Satellites, & Terra Lunam ad se alliciant viribus quadratis distantiarum reciproce proportionalibus; & quamquam motus Apogei Lunæ aberrationem tantillam ab hac lege innuere esset visus, tamen à celeberrimo Clairaut primum pulcherrimus consensus est evictus, ita ut jam audacter asseverare queamus non solum Solem sed etiam cunctos planetas vi attractrice esse prædictos, qua omnia corpora ad maximas etiam distantias remota ad se præcise secundum illam constantem legem attrahant. Quin etiam pari fere fiducia pronunciare licet, singulorum planetarum vires, quas ad distantias æquales exerunt, ipsorum massis esse proportionales, id quod communis centri gravitatis status postulare videtur: de cætero massam seu quantitatem materiæ, quam quisque planeta continet, aliunde nobis cognoscere non datur; nihilque impedit, quo minus id, cui vis absoluta cujusque planetæ revera est proportionalis, nomine massæ ejus designemus; hinc saltem nullus certe error est pertimescendus. Terra igitur perinde ac reliqui planetæ omnes non solum versus Solem, sed etiam versus singulos reliquos planetas viribus legi isti sacræ conformibus sollicitatur, à quibus sine dubio præter promotionem illam apheliorum & nodorum aliæ vehementer exiguæ perturbationes efficiuntur, quarum inventio in Aſtronomia ſine dubio maximi eſt momenti.

Hinc naſcitur quæſtio latiſſime patens ad Mechanicam referenda, qua determinatio motus plurium corporum, quæ ſe mutuo attrahant in ratione reciproca duplicata diſtantiarum, requiritur; cujus ſolutio eo ardentius eſt expetenda, quod omnia incrementa Aſtronomiæ, quæ adhuc deſiderantur, ex ea derivanda videantur. Verum enodatio hujus quæſtionis tot tantiſque difficultatibus eſt involuta, ut ſi in genere ſpectetur, vires ingenii humani longe ſuperare videatur; etſi enim caſus duorum corporum facilem habeat ſolutionem, tamen ſtatim ac tria aſſumuntur, nulla adhuc inventa ſunt artificia, quorum ope ad motus determinationem pervenire licuerit, unde multo minus pro caſu plurium corporum quicquam ſperare poſſumus. Interim tamen cum perturbationes, quas planetæ ſibi mutuo inferunt, ſint perquam exiguæ, neque eæ, quæ ab actione unius oriuntur, à reliquis affici ſint cenſendæ; hinc non contemnendum ſubſidium, impetramus aliquid ſaltem in hoc arduo negotio præſtandi, dum effectus ſingulorum planetarum ſeorſim inveſtigare licebit, & quoniam ſunt minimi, conſuetis calculi approximationibus, quarum in hujuſmodi quæſtionibus uberrimus

uberrimus ſolet eſſe uſus, totum negotium confici debebit. In hunc etiam modum Illustris Academia Regia Parisina iſtam quæſtionem tractandam judicavit, cujus præceptis ut pro viribus ſatisfaciam, operam dabo, ut primo hoc abſtruſſimum argumentum ex primis Mechanicæ fontibus dilucide evolvam; atque ad æquationes analyticas perducam: tum vero quibuſnam modis ex iis aliquid per approximationes concludi queat, accuratius inveſtigabo. Denique præcepta quæ elicuero, ad perturbationes motus terræ accommodabo, examinaturus, quantum ſingula hujus motus elementa ab actione reliquorum planetarum continuo immutentur, quod inſtitutum ſequentibus ſectionibus abſolvere conabor.

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SECTIO PRIMA.

Generalis investigatio motus corporis à viribus quibuscunque impulsi.

§. I. SUMATUR pro lubitu cum planum, ad quod motus corporis referatur, quodque plano tabulæ repræsentari concipiatur, tum vero in hoc plano linea recta fixa CA, atque in hac ipsa punctum fixum C, ubi quasi motus spectator sit constitutus. Jam ad quodvis tempus, ubicunque corpus motum versetur veluti in R, de ejus loco R in illud planum demittatur perpendiculum RQ, ita ut punctum Q ejus locum ad hoc planum relatum exhibeat, deinde etiam ex puncto fixo C ad ambo loca *Fig. I.*

R & Q ducantur rectæ CR & CQ. Quo facto perſpicuum eſt, ſi ad quodvis tempus aſſignare valeamus cum angulos ACQ & QCR, tum magnitudinem ſive rectæ CQ ſive CR, locum corporis R perfecte fore cognitum, indeque verum corporis motum innoteſcere; ita ut plena motus cognitio determinatione horum trium elementorum contineatur.

§. II. Hoc autem potiſſimum modo inveſtigationem motus inſtituo, cum quod videtur maxime naturalis, tum vero præcipue quod ad conſuetudinem Aſtronomorum, qua motus corporum cœleſtium conſiderare ſolent, imprimis eſt accommodatus. Namque ſi planum fixum tanquam planum eclipticæ ſpectemus, rectamque CA tanquam rectam ad ejus quodpiam punctum fixum directam, corpore moto in R exiſtente, angulus ACQ ejus longitudinem, angulus vero QCR ejus latitudinem referet; & quemadmodum recta CR ejus diſtantiam veram à puncto C denotat, ita recta CQ diſtantiam ejus curtatam deſignabit, quarum alteram tantum in calculum introduxiſſe ſufficiet. Vocemus igitur pro quovis tempore propoſito,

I. Longitudinem corporis ſeu angulum $ACQ = \varphi$.
II. Latitudinem ejus ſeu angulum $QCR \ldots = \psi$.
III. Diſtantiam curtatam ſeu rectam $CQ \ldots = x$.

§. III. Cognitis vero his tribus elementis, omnia, quæ ad motus notitiam pertinent, definiri poterunt. Primo enim ex diſtantia curtata $CQ = x$ & latitudine $QCR = \psi$ habetur diſtantia vera $CR = \psi$ ſeu $CR = \frac{x}{cos.\psi}$ poſito ſinu toto conſtanter $= 1$. Tum vero ipſa diſtantia corporis à plano erit $QR = x\, tang.\, \varphi$. Deinde ſi à puncto Q ad rectam fixam CA ducatur normalis QP, ex diſtantia curtata $CQ = x$ & longitudine $ACQ = \varphi$, elicitur $CP = x\, cos.\, \varphi$ & $PQ = x\, ſin.\, \varphi$; atque hoc modo pro loco puncti R, uti in Geometria fieri ſolet, ternas obtinemus coordinatas inter ſe rectangulas CP, PQ & QR. A quibus cum etiam inveſtigatio mechanica incipiat, has lineas tantiſper peculiaribus ſignis indicemus, quoad calculum ad illa primaria elementa perducere licuerit. Sit igitur $CP = x\, cos.\, \varphi = p$; $PQ = x\, ſin.\, \varphi = q$ & $QR = x\, tang.\, \psi = r$; ſicque habebimus $x = \sqrt{(pp + qq)}$; $cos.\, \varphi = \frac{p}{\sqrt{(pp+qq)}}$; $ſin.\, \varphi = \frac{q}{\sqrt{(pp+qq)}}$ & $tang.\, \psi = \frac{r}{\sqrt{(pp+qq)}}$.

§. IV. Hæc autem motus elementa ex ſollicitatione virium quarum actioni corpus fuerit ſubjectum, ſecundum præcepta mechanica determinari oportet. A quibuſcunque autem viribus corpus impellatur, eas ſemper per notam reſolutionem ad ternas directiones determinatas revocare licet. Concipiamus igitur corpus à tribus viribus ſollicitari, quarum prima urgeat ſecundum directionem RQ ad planum fixum normalem, binarum autem reliquarum directiones ſint ipſi plano parallelæ; altera quidem habeat directionem diſtantiæ curtatæ QC parallelam, altera vero huic normalem, cui in plano fixo parallela ſit recta QN ad CQ normalis. Iſtas vires ſtatuamus acceleratrices, ſive jam ad corporis maſſam applicatas, eaſque denotemus:

I. Vim acceleratricem ſecundum $QC = V$.
II. Vim acceleratricem ſecundum $QN = T$
III. Vim acceleratricem ſecundum $RQ = R$

Ita, ut, quomodo per has vires terna illa elementa φ, ψ & x determinentur, ſit inveſtigandum.

§. V. Cum autem regulæ mechanicæ ad ternas coordinatas normales, quarum directiones perpetuo maneant fixæ accommodatæ eſſe ſoleant, harum autem virium tertia tantum RQ cum una coordinatarum conveniat, dum diarum reliquarum directiones QC & QN maxime sunt variabiles, & has ad directiones fixas revocari conveniet. Dabit igitur vis V resoluta:

Secundum directionem $PC = V\,cos.\,\phi$

& secundum QP vim $= V\,sin.\,\phi$;

vis vero T simili modo resoluta:

Secundum directionem $PC = -T\,sin\,\phi$

& secundum QP vim $= T\,cos.\,\phi$.

Hinc itaque pro directionibus nostrarum ternarum coordinatarum PC, QP, & RQ obtinebimus tres vires acceleratrices sequentes quæ sunt:

I. Vis acceleratrix secundum $PC = V\,cos.\,\phi - T\,sin.\,\phi$

II. Vis acceleratrix secundum $QP = V\,sin.\,\phi + T\,cos.\,\phi$

III. Vis acceleratrix secundum $RQ = R$.

§. VI. Quoniam actio harum ternarum virium ad diminutionem coordinatarum respondentium tendit, accelerationes quæ corpori inde secundum easdem coordinatas inducuntur negativæ sunt concipiendæ. Cum igitur posito temporis elemento $= dt$, sint corporis celeritates secundum has coordinatas $\frac{dp}{dt}$; $\frac{dq}{dt}$ & $\frac{dr}{dt}$; si elementum temporis dt pro constanti assumamus, erunt ipsæ accelerationes secundum istas directiones $\frac{ddp}{dt^2}$; $\frac{ddq}{dt^2}$; $\frac{ddr}{dt^2}$, quæ viribus illis acceleratricibus negative sumtis debent esse proportionales. Proportionalitate ergo, uti fieri solet, stabilita obtinebimus ternas sequentes æquationes.

$$\text{I. } ddp = -\tfrac{1}{2}dt^2(V\,cos.\,\phi - T\,sin.\,\phi)$$
$$\text{II. } ddq = -\tfrac{1}{2}dt^2(V\,sin.\,\phi + T\,cos.\,\phi)$$
$$\text{III. } ddr = -\tfrac{1}{2}R\,dt^2$$

quarum æquationum resolutione tota motus determinatio continetur.

§. VII. Jam iterum ambas vires V & T commode à se invicem separare licet, ut pateat quid utraque seorsim præstet. Nam I × $cos.\,\phi$ + II × $sin.\,\phi$ dat

$$ddp\,cos.\,\phi + ddq\,sin.\,\phi = -\tfrac{1}{2}V\,dt^2$$

Deinde II × $cos.\,\phi$ — I × $sin.\,\phi$ præbet hanc æquationem

$$ddq\,cos.\,\phi - ddp\,sin.\,\phi = -\tfrac{1}{2}T\,dt^2.$$

at ex tertia vi R nascitur æquatio $ddr = -\frac{1}{2}R\,dt^2$. Nunc igitur recordandum est nos supra posuisse

$$p = cos.\,\phi;\quad q = x\,sin.\,\phi \;\&\; r = x\,tang.\,\psi$$

unde loco quantitatum subsidiariarum p, q, r, elementa nostra principalia ϕ, ψ & x in calculum introduci poterunt. Tres autem emergent æquationes, quæ propterea his tribus elementis definiendis sufficient: atque ita tota investigatio à principiis mechanicis ad Analysin puram traducetur.

§. VIII. Cum sit $p = x\,cos.\,\phi$ & $q = x\,sin.\,\phi$ erit differentiando;

$$dp = dx\,cos.\,\phi - x\,d\phi\,sin.\,\phi \;\&\; dq = dx\,sin.\,\phi + x\,d\phi\,cos.\,\phi$$

denuoque differentiando

$$ddp = ddx\,cos.\,\phi - 2dxd\phi\,sin.\,\phi - xd\phi^2\,cos.\,\phi - xdd\phi\,sin.\,\phi$$
$$ddq = ddx\,sin.\,\phi + 2dxd\phi\,cos.\,\phi - xd\phi^2\,sin.\,\phi + xdd\phi\,cos.\,\phi$$

unde per combinationem elicitur

$$ddp\,cos.\,\phi + ddq\,sin.\,\phi = ddx - x\,d\phi^2$$
$$\&\; ddq\,cos.\,\phi - ddp\,sin.\,\phi = 2\,dx\,d\phi + x\,dd\phi.$$

Valorem autem ipsius $r = x\,tang.\,\psi$ nulla adhibita evolutione tantisper retineamus, donec compererimus, quomodo aptissime eum tractari conveniat. Hoc itaque pacto totum negotium ad resolutionem trium sequentium æquationum erit perductum:

$$\text{I. } ddx - x\,d\phi^2 = -\tfrac{1}{2}V\,dt^2.$$
$$\text{II. } 2\,dx\,d\phi + x\,dd\phi = -\tfrac{1}{2}T\,dt^2$$
$$\text{III. } dd.\,x\,tang.\,\psi = -\tfrac{1}{2}R\,dt^2.$$

§. IX. Cum hæ æquationes ſint differentiales ſecundi gradus, temporis differentiali dt ſumto conſtante, primum diſpiciendum eſt, quamnam proportionem differentialia prima $d\varphi$, $d\psi$ & dx cum inter ſe tum ad temporis differentiale dt teneant, quod etſi ſine introductione formularum integralium fieri nequit, quamdiu vires ſollicitantes V, T & R in genere conſideramus, tamen eæ proportionem quæſitam minus perturbare ſunt cenſendæ. Quin etiam in negotio quod ſuſcipimus, ipſæ vires V, T & R quantitates incognitas, φ, ψ & x cum tempore t implicare reperientur, quominus earum ſeparatio perfecta expectari poterit. Pro initio igitur contenti eſſe debebimus, formulas noſtras à contemplatione differentialium ſecundorum liberaſſe, & quocunque modo relationem differentialium primorum determinaſſe, ut deinceps, approximationum artificio in ſubſidium vocato, ipſarum quantitatum finitarum relationem inde colligere valeamus.

§. X. Tertiam quidem æquationem $dd.\, x\, tang.\, \psi = -\frac{1}{2} R\, dt^2$ tantiſper ſeponamus, poſtmodum inveſtigaturi, quo modo ejus ratio convenientiſſime haberi queat; ambas igitur priores, quæ ſunt:

$$\text{I. } ddx - x\,d\varphi^2 = -\tfrac{1}{2} V\,dt^2 \ \&$$

$$\text{II. } 2\,dx\,d\varphi + x\,dd\varphi = -\tfrac{1}{2} T\,dt^2$$

accuratius perpendamus, ut inde relationem differentialium primorum dx, $d\varphi$ & dt eliciamus. Ac primo quidem prius membrum ſecundæ æquationis ſi per x multiplicetur, redditur integrale, proditque

$$d(xx\,d\varphi) = -\tfrac{1}{2}\textstyle\int T x\,dt^2$$

$$\text{ſeu } xx\,d\varphi = \tfrac{1}{2}\,dt\,(C - \textstyle\int T x\,dt)$$

unde ſi vis T ſecundum directionem QN trahens evaneſcat, oritur æquabilis arearum deſcriptio. Hinc autem patet eandem illam æquationem integrabilem reddi ſi multiplicetur non ſolum per x, ſed etiam inſuper per functionem

functionem quamcunque ipſius $xx\,d\varphi$; Multiplicetur ergo per $x\,d\varphi$; eritque integrale:

$$\tfrac{1}{2} x^4\,d\varphi^2 = -\tfrac{1}{2}\,dt^2 \textstyle\int T x^3\,d\varphi;$$

$$\text{ſeu } x^4\,d\varphi^2 = dt^2 (A - \textstyle\int T x^3\,d\varphi);$$

unde invenitur

$$dt^2 = \frac{x^4\,d\varphi^2}{A - \int T x^3\,d\varphi} \ \& \ dt = \frac{xx\,d\varphi}{\sqrt{(A - \int T x^3\,d\varphi)}}.$$

§. XI. Cum igitur ſit $x\,d\varphi^2 = \frac{dt^2}{x^3}(A - \int T x^3\,d\varphi)$, hoc valore in prima æquatione ſubſtituto habebimus:

$$ddx = dt^2 \left(\tfrac{A}{x^3} - \tfrac{1}{x^3}\textstyle\int T x^3\,d\varphi - \tfrac{1}{2} V\right),$$

quæ per $2\,dx$ multiplicata & integrata præbet:

$dx^2 = dt^2 \left(B - \frac{A}{xx} - 2\int \frac{dx}{x^3} \int T x^3\,d\varphi - \int V\,dx\right)$. At eſt

$$-2\textstyle\int \frac{dx}{x^3} \int T x^3\,d\varphi = \frac{1}{xx}\int T x^3\,d\varphi - \int T x\,d\varphi$$

quo valore introducto erit:

$dx^2 = dt^2 \left(B - \frac{A}{xx} + \frac{1}{xx}\int T x^3\,d\varphi - \int T x\,d\varphi - \int V\,dx\right)$, vel

$x^2\,dx^2 = dt^2 \left(Bxx - A + \int T x^3\,d\varphi - xx\int (T x\,d\varphi + V\,dx)\right)$,

unde nanciſcimur

$$dt = \frac{\pm\, x\,dx}{\sqrt{(Bxx - A + \int T x^3\,d\varphi - xx\int(Tx\,d\varphi + V\,dx))}} \ \&$$

$$d\varphi = \frac{\pm\, dx\sqrt{(A - \int T x^3\,d\varphi)}}{x\sqrt{(Bxx - A + \int T x^3\,d\varphi - xx\int(Tx\,d\varphi + V\,dx))}}.$$

§. XII. Ambiguitas ſignorum, quam motus natura involvit, ita ab arbitrio noſtro pendet, ut poſitivum valeat, ſi motum ab eo loco, ubi corpus puncto fuit proximum, definire velimus, negativum vero ſi à diſtantia maxima diſceſſerit. Quoniam igitur in Aſtronomia uſus eſt, motum corporum à maxima diſtantia computare, valeat ſignum negativum, ut habeamus has duas æquationes:

$$dt = \frac{-\,x\,dx}{\sqrt{(Bxx - A + \int T x^3\,d\varphi - xx\int(Tx\,d\varphi + V\,dx))}}$$

$$d\varphi = \frac{-\,dx\sqrt{(A - \int T x^3\,d\varphi)}}{x\sqrt{(Bxx - A + \int T x^3\,d\varphi - xx\int(Tx\,d\varphi + V\,dx))}};$$

cujus poſterioris loco & hæc primum inventa $d\varphi = \frac{dt}{xx} \surd (A - \int T x^3 d\varphi)$ uſurpari poteſt. Sunt autem A & B quantitates conſtantes, per duplicem integrationem invectæ, quæ deinceps ad quemvis caſum oblatæ accommodari debent.

§. XIII. Quando vis normalis T evaneſcit, alteraque vis V ad C tendens per ſolam diſtantiam x determinatur, utraque æquatio habebit variabiles ſeparatas, ita ut non ſolum differentialium dt & $d\varphi$ ratio ad dx abſolute poſſit aſſignari, ſed etiam per integrationem ipſæ quantitates finitæ t & φ per diſtantiam x definiri: hocque ergo caſu problematis ſolutio perfecta poterit exhiberi. Neque vero in genere hæ formulæ magis ad uſum accommodari poſſe videntur. Sed contentos nos eſſe opportet, hoc pacto rationem differentialium dt, $d\varphi$ & dx elicuiſſe. Quamvis enim adſint formulæ integrales $\int T x d\varphi$, $\int T x^3 d\varphi$, & $\int V dx$ hanc ipſam rationem involventes, eæ tamen negotium approximationis non multum turbant, dummodo earum valores ſint perquam exigui, propterea quod tunc ſufficit rationes differentialium prope veras noſſe. Verum ipſum approximationis negotium alias requirit conſiderationes, antequam cum ſucceſſu ſuſcipi queat, quas deinceps evolvemus.

§. XIV. Perductis igitur binis prioribus æquationibus differentio-differentialibus ad formulas ſimpliciter differentiales, quæ ad uſum maxime videntur accommodatæ, tertiam quoque æquationem $ddr = -\frac{1}{2} R dt^2$ inſtituto convenientius transformare conemur. Quæ cum latitudinis ψ determinationem ob $r = x\, tang. \psi$ contineat commodiſſime ea inſtituetur, ſi more apud Aſtronomos recepto lineam nodorum cum inclinatione orbitæ ad planum aſſumtum in calculum introducamus. Hunc in finem conſideretur quovis momento planum, quod puncto fixo C & ſpatiolo à corpore jam jam percurrendo

determinetur. Quodque pro iſto ſaltem momento planum orbitæ appellare liceat. Sit igitur dum corpus verſatur in R recta $C\Omega$ interſectio plani orbitæ & plani aſſumti, quæ linea nodorum vocari ſolet, atque ad latitudinem ψ commodius inveſtigandam ponamus:

I. Longitudinem lineæ nodorum ſeu angulum $AC\Omega = \pi$,
II. Inclinationem plani orbitæ ad planum aſſumtum $= G$;

quæ duo nova elementa tanquam utcunque variabilia contemplor.

§. XV. Ad inclinationem autem definiendam ex punctis R & Q ad lineam nodorum $C\Omega$ ducantur normales RS & QS, quarum inclinatio ſeu angulus QSR inclinationem metietur, ita ut ſit $QSR = G$. Deinde ob angulum $\Omega CQ = \varphi - \pi$ & $CQ = x$, erit $QS = x\, ſin. (\varphi - \pi)$; unde fit $QR = x\, ſin. (\varphi - \pi)\, tang. G$. Cum igitur habeamus $QR = r = x\, tang\, \psi$, erit:

$$tang. \psi = ſin.\ (\varphi - \pi)\ tang.\ G;$$

ſicque ex elementis φ, π & G, latitudo quæſita ψ reperietur. Quoniam autem loco latitudinis ψ duo nova elementa π & G æque ad locum corporis ſequentem pertineant; unde differentiale ipſius ψ ſeu $tang. \psi$ idem prodire debet ſive elementa ambo π & G ſumantur conſtantia, ſive ambo variabilia, ex qua proprietate relatio inter π & G innoteſcet, quæ locum quartæ æquationis ſuſtinebit, ſi quidem jam quatuor elementa x, φ, π & G in calculo habemus.

§. XVI. Cum igitur poſitis π & G conſtantibus ſit

$$d.\ tang.\ \psi = d\varphi\ coſ.\ (\varphi - \pi)\ tang.\ G,$$

iiſdem autem tanquam variabiles tractatis prodeat

$$d.\, tang. \psi = (d\varphi - d\pi)\, coſ. (\varphi - \pi)\, tang. G + ſin. (\varphi - \pi)\, d.\, tang. G;$$

his valoribus inter ſe æquatis obtinebimus

$$d.\,tang.\,G = \frac{d\pi\, cof.\,(\varphi-\pi)}{fin.\,(\varphi-\pi)}\, tang.\,G,$$

$$\text{feu}\ \frac{d.\,tang.\,G}{tang.\,G} = \frac{d\pi\, cof.\,(\varphi-\pi)}{fin.\,(\varphi-\pi)}.$$

Sufficit igitur longitudinem nodi π ejufque variationem determinaviffe, indeque facile inclinatio orbitæ G definietur; eft enim $\frac{d.\,tang.\,G.}{tang.\,G.}$ differentiale logarithmi ipfius G, quod fi ita indicemus $d.\,l\,tang.\,G$, erit

$$d.\,l\,tang.\,G = \frac{d\pi\, cof.\,(\varphi-\pi)}{fin.\,(\varphi-\pi)} = d\pi\, cot.\,(\varphi-\pi).$$

Patet igitur nifi linea nodorum fit fixa ideoque $d\pi = 0$, inclinationem orbitæ continuis variationibus effe obnoxiam, quarum autem alteræ ex alteris facile definiri poterunt.

§. XVII. Propofitum autem eft invenire ddr, cujus valor ob $r = x\,tang.\,\psi$ eft: $ddx\,tang.\,\psi + 2dx\,d.\,tang.\,\psi + x\,dd.\,tang.\,\psi = -\frac{1}{2}R\,dt^2$. Verum invenimus:

$$tang.\,\psi = fin.\,(\varphi-\pi)\ tang.\,G\text{, atque}$$

$$d.\,tang.\,\psi = d\varphi\, cof.\,(\varphi-\pi)\ tang.\,G\text{; unde ob}$$

$$d.\,tang.\,G = \frac{d\pi\, cof.\,(\varphi-\pi)}{fin.\,(\varphi-\pi)}\, tang.\,G,$$ erit porro differentiando

$$dd.\,tang.\,\psi = dd\varphi\, cof.\,(\varphi-\pi)\, tang.\,G - d\varphi\,(d\varphi - d\pi).$$

$$fin.\,(\varphi-\pi)\ tang.\,G + \frac{d\varphi\, d\pi\, cof.\,(\varphi-\pi)^2}{fin.\,(\varphi-\pi)}\, tang.\,G,$$

$$\text{five}\ dd.\,tang.\,\psi = dd\varphi\left(cof.\,(\varphi-\pi) - d\varphi^2 fin.\,(\varphi-\pi) + \frac{d\varphi\, d\pi}{fin.\,(\varphi-\pi)}\right) tang.\,G.$$

Quibus valoribus fuftitutis tertia æquatio induet hanc formam:

$$\left\{\begin{matrix} +\,ddx\, fin.\,(\varphi-\pi) + 2\,dx\,d\varphi\, cof.\,(\varphi-\pi) \\ +\,x\,dd\varphi\, cof.\,(\varphi-\pi) - x\,d\varphi^2 fin.\,(\varphi-\pi) + \dfrac{x\,d\varphi\, d\pi}{fin.\,(\varphi-\pi)} \end{matrix}\right\} tang.\,G = -\tfrac{1}{2}R\,dt^2$$

At ex binis prioribus æquationibus erat: $ddx - x\,d\varphi^2 = -\frac{1}{2}V\,dt^2$ & $2\,dx\,d\varphi + x\,dd\varphi = -\frac{1}{2}T\,dt^2$; ficque fiet:

$$\left(-\tfrac{1}{2}V\,dt^2 fin.\,(\varphi-\pi) - \tfrac{1}{2}T\,dt^2 cof.\,(\varphi-\pi) + \frac{x\,d\varphi\, d\pi}{fin.\,(\varphi-\pi)}\right) tang.\,G = -\tfrac{1}{2}R\,dt^2\text{, feu}$$

$$\frac{x\,d\varphi\, d\pi}{fin.\,(\varphi-\pi)} = \tfrac{1}{2}dt^2\left(V fin.\,(\varphi-\pi) + T cof.\,(\varphi-\pi) - \frac{R}{tang.\,G}\right).$$

§. XVIII. Ex viribus igitur follicitantibus V, T & R quaterna noftra elementa x, φ, π & G ad quodvis tempus t per fequentes quaternas æquationes differentiales primi gradus determinantur.

$$\text{I.}\ dt = \frac{-x\,dx}{\sqrt{(Bxx - A + \int Tx^3 d\varphi - xx\int(Tx\,d\varphi + V\,dx))}}$$

$$\text{II.}\ d\varphi = \frac{dt}{xx}\sqrt{(A - \int Tx^3 d\varphi)}.$$

$$\text{III.}\ d\pi = \frac{dt^2 fin.\,(\varphi-\pi)}{2\,x\,d\varphi}\left(V fin.\,(\varphi-\pi) + T cof.\,(\varphi-\pi) - \frac{R}{tang.\,G}\right).$$

$$\text{IV.}\ d.\,l.\,tang.\,G = \frac{d\pi\, cof.\,(\varphi-\pi)}{fin.\,(\varphi-\pi)}\text{, feu }d.\,l.\,tang.\,G = \frac{dt^2 cof.\,(\varphi-\pi)}{2\,x\,d\varphi}\left(V fin.\,(\varphi-\pi) + T cof.\,(\varphi-\pi) - \frac{R}{tang.\,G}\right).$$

quæ æquationes in genere vix tractabiliores reddi poffe videntur, nifi certa quædam virium follicitantium relatio accedat.

SECTIO II.

Reductio harum formularum ad caſum quo corpus imprimis ad punctum fixum urgetur vi in quadratis diſtantiarum reciproce proportionali, cujus reſpectu reliquæ vires ſunt valde parvæ.

§. XIX. QUONIAM nobis eſt propoſitum in perturbationes motus planetarum quatenus ab eorum actione mutua oriuntur inquirere, tantum jam pro certo aſſumere licet, eorum motum, proxime ſaltem, regulis Keplerianis eſſe conformem ac perturbationes quas definiri oportet, vehementer eſſe exiguas. Moderatio igitur præcipua motus eorum efficitur à vi quadam ad punctum fixum ſecundum rationem reciprocam duplicatam diſtantiarum tendente, præ qua reliquæ vires quaſi evaneſcant. Diſerte enim fateri cogor, niſi hujuſmodi vis inter reliquas vires ſollicitantes longe emineat, nullo plane modo me perſpicere, qua ratione ad aliqualem ſaltem motus cognitionem pertingere nobis liceat. Minime autem caſui tribuendum videtur, quod hujuſmodi motus, quorum inveſtigatio vires noſtras penitus ſuperaret, etiamſi æque facile exiſtere potuiſſent, in mundo non deprehendantur.

FIG. I. §. XX. Sit igitur C id punctum, ad quod vis illa principalis quadratis diſtantiarum reciproce proportionalis dirigatur, ſi quidem hactenus hoc punctum ab arbitrio noſtro pendebat. Neque tamen hanc vim quadrato diſtantiæ veræ RC reciproce proportionalem ſtatuamus quoniam ejus reductio ad noſtras formulas denuo

latitudinem involveret. Sed quoniam planum fixum ſemper ita aſſumere licet, ut corpus ab eo non niſi quam minime recedat, anguluſque ψ perpetuus ſit valde exiguus, caſum ita ſtabiliamus, ut vis V ſecundum directionem QC ſollicitans habeat partem eximiam quadrato diſtantiæ $QC = x$ reciproce proportionalem, cujus reſpectu tam reliqua pars quam binæ reliquæ vires T & R pro minimis haberi queant. Statuamus igitur $V = \frac{ff}{xx} + S$ ita ut vires S, T & R præ vi $\frac{ff}{xx}$ quaſi evaneſcant.

§. XXI. Poſito autem $V = \frac{ff}{xx} + S$ erit $\int V\,dx = \frac{-ff}{x} + \int S\,dx$ & $xx\int V\,dx = -ffx + xx\int S\,dx$, quo valore in noſtris formulis ſurrogato habebimus,

$$\text{I. } dt = \frac{-x\,dx}{\sqrt{(Bxx + ffx - A + \int Tx^3\,d\varphi - xx(\int Tx\,d\varphi + \int S\,dx))}}.$$

$$\text{II. } d\varphi = \frac{dt}{xx}\sqrt{(A - \int Tx^3\,d\varphi)}, \text{ ſeu } d\varphi = \frac{-dx(A - \int Tx^3\,d\varphi)}{x\sqrt{(Bxx + ffx - A + \int Tx^3\,d\varphi - xx(\int Tx\,d\varphi + \int S\,dx))}}.$$

$$\text{III. } d\pi = \frac{dt^2\,\text{ſin.}(\varphi - \pi)}{2x\,d\varphi} \times \left(\frac{xx}{ff}\text{ſin.}(\varphi - \pi) + S\,\text{ſin.}(\varphi - \pi) + T\,\text{coſ.}(\varphi - \pi) - \frac{R}{\text{tang.}\,G}\right).$$

$$\text{IV. } d.l.\,\text{tang.}\,G = \frac{d\pi\,\text{coſ.}(\varphi - \pi)}{\text{ſin.}(\varphi - \pi)}, \text{ ſeu } d.l.\,\text{tang.}\,G = \frac{dt^2\,\text{coſ.}(\varphi - \pi)}{2x\,d\varphi} \times \left(\frac{ff}{xx}\text{ſin.}(\varphi - \pi) + S\,\text{ſin.}(\varphi - \pi) + T\,\text{coſ.}(\varphi - \pi) - \frac{R}{\text{tang.}\,G}\right).$$

Unde latitudo ψ ita determinatur ut ſit $\text{tang.}\,\psi = \text{ſin.}(\varphi - \pi)\,\text{tang.}\,G$. Haſque æquationes, quatenus termini litteras S, T & R involventes ſunt minimi, ad inſtitutum noſtrum propius accommodari oportet.

§. XXII. Quo rationem parvitatis virium T & S facilius in calculum introducere queamus, contemplemur primum caſum, quo iſtæ vires penitus evaneſcunt; binæque priores æquationes ſequentem induent formam:

$$\text{I. } dt = \frac{-x\,dx}{\sqrt{(Bxx + ffx - A)}} \ \&$$

$$\text{II. } d\varphi = \frac{dt\sqrt{A}}{xx} = \frac{-dx\sqrt{A}}{x\sqrt{(Bxx + ffx - A)}};$$

quarum evolutio ita eſt in promtu, ut introducendo quodam angulo v, qui anomalia vera vocari ſolet, ſi ponatur $x = \frac{b}{1 - k\cos. v}$, conſtantes illæ A & B per has novas b & k ita definiri queant, ut formula irrationalis $\sqrt{(Bxx + ffx - A)}$ evaneſcat ſive angulus v ſit $= o$ ſive duobus rectis æqualis. Atque hinc oritur notiſſima motus elliptici ratio, pro quo littera b denotat ſemiparametrum ellipſis & k ejus excentricitatem, ſeu focorum diſtantiam per axem tranſverſum diviſam. Accedentibus autem viribus minimis T & S motus aliquantillum ab hac lege diſcrepabit.

§. XXIII. Diſcrimen ſcilicet in hoc conſiſtet, quod jam quantitates b & k non amplius futuræ ſunt conſtantes, ſed variabilitatem à viribus T & S oriundam implicent. Quamobrem ponamus $x = \frac{p}{1 - q\cos. v}$, ſitque ut ante v ejuſmodi angulus, quo ſive evaneſcente ſive ad 180° increſcente, diſtantia x fiat ſive maxima ſive minima: ſeu quod eodem redit ut fiat $dx = o$, ſi ſit $\sin. v = o$. Cum igitur ſit: $dx = -\frac{dt}{x} \times \sqrt{(Bxx + ffx - A + \int Tx^3 d\varphi - xx(\int Tx\,d\varphi + \int S\,dx))}$ formula irrationalis, poſito $x = \frac{p}{1 - q\cos. v}$, factorem $\sin. v$ involvere ſeu hujuſmodi formam $W\sin. v$ habere debebit: quod ut eveniat ipſæ quantitates variabiles p & q debito modo definiri conveniet. At poſito præterquam in

in formulis integralibus $x = \frac{p}{1 - q\cos. v}$, habebimus: $dx = -\frac{dt}{p} \times$
$\sqrt{(Bpp + ffp(1 - q\cos. v) - A(1 - q\cos. v)^2 + (1 - q\cos. v)^2\int Tx^3 d\varphi - pp(\int Tx\,d\varphi + \int S\,dx))}$.

§. XXIV. Evolvamus hanc formulam ſecundum $\cos. v$ hoc modo:

$$dx = -\frac{dt}{p}\sqrt{\left\{\begin{array}{l} Bpp + ffp - A + \int Tx^3 d\varphi - pp(\int Tx\,d\varphi + \int S\,dx) \\ - ffpq\cos. v + 2Aq\cos. v - 2q\cos. v.\int Tx^3 d\varphi \\ - Aqq\cos. v^2 + qq\cos. v^2 \int Tx^3 d\varphi \end{array}\right\}}$$

jam ut in ſigno radicali $\sin. v^2$ ſeu $1 - \cos. v^2$ involvatur, reddamus terminos ipſum $\cos. v$ continentes nihilo æquales, unde diviſione per $2q\cos. v$ inſtituta fit:

$$-\tfrac{1}{2}ffp + A - \int Tx^3 d\varphi = o, \text{ ſeu } A - \int Tx^3 d\varphi = \tfrac{1}{2}ffp,$$

hocque valore ſubſtituto orietur:

$$dx = -\frac{dt}{p}\sqrt{\left\{\begin{array}{l} Bpp + \tfrac{1}{2}ffp - pp(\int Tx\,d\varphi + \int S\,dx) \\ - \tfrac{1}{2}ffpqq\cos. v^2. \end{array}\right\}}$$

Fiat porro:

$$Bpp + \tfrac{1}{2}ffp - pp(\int Tx\,d\varphi + \int S\,dx) = \tfrac{1}{2}ffpqq,$$

ſeu $\tfrac{1}{2}ffqq = Bp + \tfrac{1}{2}ff - p(\int Tx\,d\varphi + \int S\,dx)$
quo facto habebitur:

$$dx = -\frac{dt}{p}\sqrt{(\tfrac{1}{2}ffpqq - \tfrac{1}{2}ffpqq\cos. v^2)} = -\frac{fq\,dt\sin. v}{\sqrt{2p}}.$$

§. XXV. Poſito ergo $x = \frac{p}{1 - q\cos. v}$, ut p exprimat ſemiparametrum, & q excentricitatem ellipſis, utramque ob perturbationes variabilem, atque v anomaliam veram, hæ quantitates ita per vires T & S determinari debent ut ſit:

$$p = \frac{2A - 2\int Tx^3 d\varphi}{ff}, \ \&$$

$$qq = \frac{ff + 2Bp - 2p(\int Tx\,d\varphi + \int S\,dx)}{ff}.$$

Cum autem hæ ipsæ quantitates, evanescentibus viribus T & S, evadant constantes, sicque earum valores quasi medii prodire debeant, ponantur hi $p = b$ & $qq = kk$, hincque constantes A & B instituto convenienter ita definientur ut sit:

$$b = \frac{2A}{ff} \text{ seu } A = \tfrac{1}{2} b ff, \ \&$$
$$kk = \frac{ff + 2Bb}{ff} \text{ seu } B = \frac{-ff(1-kk)}{2b}.$$

Hinc itaque habebimus:

$$p = b - \frac{2}{ff} \int T x^3 \, d\varphi,$$
$$qq = 1 - \frac{(1-kk)p}{b} - \frac{2p}{ff}\left(\int T x \, d\varphi + \int S \, dx\right).$$

§. XXVI. Valoribus igitur p & q ita stabilitis, ut earum variabilitas tantum à viribus T & S, quarum actio est valde parva, pendeat, obtinebimus inde:

Ipsam distantiam curtatam $x = \frac{p}{1 - q \cos. v}$,
Ejusque differentiale . . . $dx = \frac{-fq \, dt \sin. v}{\sqrt{2p}}$.

Tum vero erit ut ante invenimus:

$$d\varphi = \frac{dt}{xx}\sqrt{\left(A - \int T x^3 d\varphi\right)} = \frac{dt}{xx}\sqrt{\left(\tfrac{1}{2} b ff - \int T x^3 d\varphi\right)},$$

vel etiam $d\varphi = \frac{f dt}{xx}\sqrt{\tfrac{1}{2}p} = \frac{f dt (1 - \cos. v)^2}{p\sqrt{2p}}$.

Sicque commodius differentialia dx & $d\varphi$ per differentiale temporis dt habentur expressa. Hæ autem formulæ in locum binarum priorum æquationum (§. XXI) inventarum sunt substituendæ; quod vero ad binas posteriores attinet, quæ ad latitudinem spectant, eas deinceps seorsim considerabo, quia earum evolutio non tantis est subjecta difficultatibus. Hic igitur ita binis prioribus inhærebo, quasi binæ posteriores prorsus abessent.

§. XXVII. Verum conditio, quâ esse debet $dx = \frac{-fq \, dt \sin. v}{\sqrt{2p}}$ existente $x = \frac{p}{1 - q \cos. v}$, novam determina-

tionem continet, quæ indolem anomaliæ veræ v & quemadmodum ejus differentiale sit comparatum definiet. Cum enim sit $1 - q \cos. v. = \frac{p}{x}$ erit, differentiando & pro dx valorem $-\frac{fq \, dt \sin. v}{\sqrt{2p}}$ substituendo:

$$-dq \cos. v + q \, dv \sin. v = \frac{dp}{x} + \frac{fq \, dt \sin. v}{xx}\sqrt{\tfrac{1}{2}p}, \text{ seu}$$

$$q \, dv \sin. v. = \frac{dp}{x} + dq \cos. v + \frac{fq \, dt \sin. v}{xx}\sqrt{\tfrac{1}{2}p}.$$

Per valores autem pro p & q supra inventos, etiam harum quantitatum differentialia innotescunt: erit enim

$$dp = -\frac{2Tx^3 d\varphi}{ff} = -\frac{Tx \, dt}{f}\sqrt{2p} \text{ ob } d\varphi = \frac{f \, dt}{xx}\sqrt{\tfrac{1}{2}p}, \ \&$$

$$d.\frac{qq}{p} = -\frac{dp}{pp} - \frac{2Tx \, d\varphi}{ff} - \frac{2S \, dx}{ff} =$$
$$\frac{Tx \, dt}{fpp}\sqrt{2p} - \frac{T \, dt}{fx}\sqrt{2p} + \frac{2Sq \, dt \sin. v}{f\sqrt{2p}}$$

substitutis pro dp, $d\varphi$ & dx valoribus inventis. Qua formula porro evoluta oritur:

$$\frac{2q \, dq}{p} = -\frac{Tqqx \, dt}{fpp}\sqrt{2p} + \frac{Tx \, dt}{fpp}\sqrt{2p} - \frac{T \, dt}{fx}\sqrt{2p} + \frac{2Sq \, dt \sin. v.}{f\sqrt{2p}}$$

quæ ob $p = x(1 - q \cos. v)$ reducitur ad hanc formam:

$$dq = \frac{Tx \, dt}{2fp}(2\cos. v - q - q\cos. v^2)\sqrt{2p} + \frac{Sp \, dt \sin. v}{f\sqrt{2p}}.$$

§. XXVIII. Cum igitur sit:

$$\frac{dp}{x} = -\frac{T \, dt}{f}\sqrt{2p} = -\frac{2Tx \, dt(1 - q\cos. v)}{2fp}\sqrt{2p} \text{ erit}$$

$$\frac{dp}{x} + dq \cos. v = \frac{Tx \, dt}{2fp}(2\cos. v^2 + q\cos. v - q\cos. v^3 - 2)\sqrt{2p} + \frac{Sp \, dt \sin. v \cos. v}{f\sqrt{2p}},$$

hæc forma ob $1 - \cos. v^2 = \sin. v^2$ perducitur ad istam

$$\frac{dp}{x} + dq \cos. v = -\frac{Tx \, dt \sin. v^2}{f\sqrt{2p}}(2 - q\cos. v) + \frac{Sp \, dt \sin. v \cos. v}{f\sqrt{2p}},$$

quo valore in ſuperiori expreſſione pro $q\,dv\,ſin.\,v$ inventa ſubſtituto, diviſione facta per $q\,ſin.\,v$, habebitur:

$$dv = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p} - \frac{Tx\,dt\,ſin.\,v}{fq\sqrt{2p}}(2 - q\,coſ.\,v) + \frac{Sp\,dt\,coſ.\,v}{fq\sqrt{2p}}.$$

Hæc autem porro, ob $x(1 - q\,coſ.\,v) = p$, tranſit in formas ſequentes:

$$dv = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p} - \frac{T\,dt\,ſin.\,v}{fq}\sqrt{2p} - \frac{Tx\,dt\,ſin.\,v.\,coſ.\,v.}{f\sqrt{2p}} + \frac{Sp\,dt\,coſ.\,v}{fq\sqrt{2p}},\ \text{ſeu}$$

$$dv = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p} - \frac{p\,dt(2T\,ſin.\,v - S\,coſ.\,v)}{fq\sqrt{2p}} - \frac{Tx\,dt\,ſin.\,v\,coſ.\,v}{f\sqrt{2p}},\ \text{vel etiam}$$

$$dv = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p} + \frac{x\,dt(S\,coſ.\,v.(1 - q\,coſ.\,v) - T\,ſin.\,v.(2 - q\,coſ.\,v))}{fq\sqrt{2p}}.$$

§. XXIX. Quoniam porro eſt:

$$d\varphi = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p}\ \&\ dx = -\frac{fq\,dt\,ſin.\,v}{\sqrt{2p}}\ \text{erit:}$$

$$\int Tx^3\,d\varphi = f\int Tx\,dt\sqrt{\tfrac{1}{2}p}$$

$$\int Tx\,d\varphi = f\int \frac{T\,dt}{x}\sqrt{\tfrac{1}{2}p}\ \&$$

$$\int S\,dx = -f\int \frac{Sq\,dt\,ſin.\,v}{\sqrt{2p}},\ \text{ideoque}$$

$$\int Tx\,d\varphi + \int S\,dx = f\int \frac{dt}{\sqrt{2p}}(T(1 - q\,coſ.\,v) - Sq\,ſin.\,v).$$

His igitur valoribus ſubſtituendis non ſolum formulæ integrales, quæ etiamnunc in calculum ingrediuntur, ad differentiale temporis reducuntur, ſed etiam omnium quantitatum variabilium, quibus jam erit utendum, differentialia per idem temporis differentiale dt erunt expreſſa. Neque vero adhuc ulla approximatione ſumus uſi, unde hæ determinationes etiam locum habent, tametſi forte vires T & S non fuerint adeo exiguæ. Interim tamen parum ſubſidii inde conſequi licet, niſi iſtæ vires valde fuerint parvæ.

§. XXX. Hæ autem formulæ maxime videntur idoneæ, ad motus aberrationes à regulis Kepplerianis definiendas; referuntur enim ad motum in ellipſi continuo

variabili tam ratione ejus parametri & excentricitatis quam ſitus lineæ abſidum. Quovis enim tempore minimo motus corporis ita conſiderari poteſt, quaſi fieret in ellipſi ſecundum regulas Keppleri, ac ſi pro quolibet tempore conſtet magnitudo iſtius ellipſis, ejuſque excentricitas una cum ſitu lineæ abſidum, ex formulis inventis verus corporis locus, quatenus ad planum aſſumtum refertur, aſſignari poterit. Aſſumo igitur ad tempus propoſitum t orbitæ ad planum noſtrum fixum relatæ eſſe

I. Semiparametrum orbitæ $= p$,

II. Excentricitatem ejus $= q$,

III. Atque anomaliam veram corporis $= v$.

Unde cum longitudo corporis poſita ſit $= \varphi$, longitudo abſidis ſummæ definietur angulo $= \varphi - v$.

§. XXXI. Ex his igitur elementis ellipticis primo deducitur diſtantia curtata $CQ = x$ ope formulæ $x = \frac{p}{1 - q\,coſ.\,v}$; unde poſito angulo $v = 0$ colligitur diſtantia abſidis ſummæ $= \frac{p}{1-q}$ & poſito $v = 180°$ diſtantia abſidis imæ à puncto $C = \frac{p}{1+q}$, quarum ſumma $\frac{2p}{1-qq}$ præbet axem tranſverſum orbitæ ellipticæ, cujus propterea ſemiſſis eſt $= \frac{p}{1-qq}$, & ſemiſſis diſtantiæ focorum $= \frac{pq}{1-qq}$: tum vero ſemi axis conjugatus erit $= \frac{p}{\sqrt{(1-qq)}}$. Deinde vero ipſa corporis longitudo ſeu angulus $ACQ = \varphi$ ita per hæc elementa determinatur, ut ſit:

$$d\varphi = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p},\ \text{ſeu}\ d\varphi = \frac{f\,dt(1 - q\,coſ.\,v)^2}{p\sqrt{2p}};$$

unde ea per integrationem elici poterit, dummodo lex conſtet, qua quantitates variabiles p, q & v cum tempore mutentur, hanc autem variabilitatis legem jam eruimus.

§. XXXII. Si nullæ adessent vires perturbantes T & S, tam parameter orbitæ $2p$ quam excentricitas q essent quantitates constantes, atque anomalia vera v cum longitudine φ paria caperet incrementa; sicque ob $d\varphi - dv = o$, linea absidum immota maneret. Videamus igitur quales mutationes hæ quantitates subire debeant acturesque cedentibus istis viribus perturbatricibus T & S. Ac primo quidem ob $Tx^3 d\varphi = fTx\,dt\sqrt{\tfrac{1}{2}p}$, tempusculo dt semiparametri p incrementum inventum est:

$$dp = -\frac{Tx\,dt}{f}\sqrt{2p} = -\frac{Tp\,dt\sqrt{2p}}{f(1-q\,cos.\,v)}$$

quod ergo tantum à vi T pendet, nisi quatenus variabilitas quantitatum q & v simul alteram vim S involvit. Hinc igitur erit:

$$\frac{dp}{p\sqrt{p}} = -\frac{T\,dt\sqrt{2}}{f(1-q\,cos.\,v)} \;\&\; \frac{2}{\sqrt{p}} = \int\frac{T\,dt\sqrt{2}}{f(1-q\,cos.\,v)}$$

seu cum medius valor ipsius p sit $= b$ erit:

$$\frac{1}{\sqrt{p}} = \frac{1}{\sqrt{b}} + \frac{1}{f\sqrt{2}}\int\frac{T\,dt}{1-q\,cos.\,v}.$$

§. XXXIII. Secundo incrementum excentricitatis dq ita expressum invenimus pro tempusculo dt, ut sit:

$$dq = \frac{Tx\,dt(2\,cos.\,v - q - q\,cos.\,v^2) + Sp\,dt\,sin.\,v}{f\sqrt{2p}},$$

quo substituto pro x valore $\frac{p}{1-q\,cos.\,v}$ abit in

$$dq = \frac{p\,dt}{f\sqrt{2p}}\left(\frac{T}{1-q\,cos.\,v}(2\,cos.\,v - q - q\,cos.\,v^2) + S\,sin.\,v\right), \text{ seu}$$

$$dq = \frac{dt}{f}\left(T\,cos.\,v + S\,sin.\,v + T.\frac{cos.\,v - q}{1-q\,cos.\,v}\right)\sqrt{\tfrac{1}{2}p}, \text{ vel etiam}$$

$$dq = \frac{dt}{f}\left(2\,T\,cos.\,v + S\,sin.\,v - \frac{Tq\,sin.\,v^2}{1-q\,cos.\,v}\right)\sqrt{\tfrac{1}{2}p};$$

in qua formula ultimus terminus præ binis præcedenti-

bus erit valde parvus, si quidem excentricitas q non adeo fuerit notabilis. Cognitis igitur viribus T & S cum anomalia vera, hinc facile colligitur, quantum excentricitas intervallo minimi tempusculi dt immutetur, quemadmodum ex formula præcedente variatio semiparametri p inotescit.

§. XXXIV. Hinc etiam definiri potest variabilitas axis transversi ellipsis, cum enim ejus semissis sit $= \frac{p}{1-qq}$ erit ejus differentiale:

$$d.\frac{p}{1-qq} = \frac{(1-qq)\,dp + 2pq\,dq}{(1-qq)^2}.$$

Quod si jam valores pro dp & dq inventi substituantur, reperitur reductione rite facta:

$$d.\frac{p}{1-qq} = -\frac{p\,dt\sqrt{2p}}{f(1-qq)^2}\,(T - q\,(T\,cos.\,v + S\,sin.\,v)).$$

Quare si semi-axis transversus ponatur $= r$, ut sit $r = \frac{p}{1-qq}$, ob $\frac{p\sqrt{2p}}{(1-qq)^2} = rr\sqrt{\frac{2}{p}}$, fiet

$$dr = -\frac{rr\,dt}{f\sqrt{\frac{1}{2}p}}\,(T - q\,(T\,cos.\,v + S\,sin.\,v)), \text{ seu}$$

$$\frac{-dr}{rr} = d.\frac{1}{r} = \frac{dt\,(T - q\,(T\,cos.\,v + S\,sin.\,v))}{f\sqrt{\frac{1}{2}p}}.$$

Unde, si semi-axis transversus medius ponatur $= a$, fiet

$$\frac{1}{r} = \frac{1}{a} + \int\frac{dt}{f\sqrt{\frac{1}{2}p}}\,(T - q\,(T\,cos.\,v + S\,sin.\,v)).$$

§. XXXV. Denique mutatio instantanea lineæ absidum est indaganda, cujus longitudo cum sit $= \varphi - v$, erit ejus incrementum tempusculo dt ortum $= d\varphi - dv$. Verum invenimus:

$$d\varphi = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p} = \frac{f\,dt\,(1-q\,cos.\,v)^2}{p\sqrt{2p}}, \;\&$$

$$dv = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p} + \frac{x\,dt(S\,cos.\,v\,(1-q\,cos.\,v) - T\,sin.\,v\,(2-q\,cos.\,v))}{fq\sqrt{2p}}, \text{ seu}$$

$$d\nu = d\varphi + \frac{dt}{fq}\left(S\,\mathrm{cof.}\nu - \frac{T\,(2 - q\,\mathrm{cof.}\,\nu)\,\mathrm{fin.}\,\nu}{1 - q\,\mathrm{cof.}\,\nu}\right)\sqrt{\tfrac{1}{2}p}.$$

Unde colligimus

$$d\varphi - d\nu = \frac{dt}{fq}\left(\frac{T\,(2 - q\,\mathrm{cof.}\,\nu)\,\mathrm{fin.}\,\nu}{1 - q\,\mathrm{cof.}\,\nu} - S\,\mathrm{cof.}\,\nu\right)\sqrt{\tfrac{1}{2}p} =$$

$$\frac{dt}{fq}\left(\frac{T\,\mathrm{fin.}\,\nu}{1 - q\,\mathrm{cof.}\,\nu} + T\,\mathrm{fin.}\,\nu - S\,\mathrm{cof.}\,\nu\right)\sqrt{\tfrac{1}{2}p},$$ five

$$d\varphi - d\nu = \frac{dt}{fq}\left(2\,T\,\mathrm{fin.}\,\nu - S\,\mathrm{cof.}\,\nu + \frac{Tq\,\mathrm{fin.}\,\nu\,\mathrm{cof.}\,\nu}{1 - q\,\mathrm{cof.}\,\nu}\right)\sqrt{\tfrac{1}{2}p}.$$

Hinc igitur patet motum lineæ abſidum eo fieri notabiliorem, quo minor fuerit excentricitas q; qua evaneſcente etiam in infinitum abire videtur. Verum notandum eſt, quo minor fuerit excentricitas, eo minus referre verum lineæ abſidum locum noſſe.

§. XXXVI. Poſtremo ad eadem elementa reducere poterimus æquationum principalium (§. XXI) binas poſteriores; cum enim ſit $d\varphi = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p}$ erit $\frac{dt^2}{2x\,d\varphi} = \frac{x\,dt}{f\sqrt{2p}}$, mutationes momentaneæ, quas cum linea nodorum tum inclinatio orbitæ ad planum fixum ſubibunt, ita per tempuſculum minimum dt erunt expreſſæ:

$$d\pi = \frac{x\,dt\,\mathrm{fin.}\,(\varphi - \pi)}{f\sqrt{2p}}\left(\frac{ff}{xx}\,\mathrm{fin.}\,(\varphi - \pi) + S\,\mathrm{fin.}\,(\varphi - \pi) + T\,\mathrm{cof.}\,(\varphi - \pi) - \frac{R}{\mathrm{tang.}\,G}\right)$$

$$d\,l\,\mathrm{tang.}\,G = \frac{x\,dt\,\mathrm{cof.}\,(\varphi - \pi)}{f\sqrt{2p}}\left(\frac{ff}{xx}\,\mathrm{fin.}\,(\varphi - \pi) + S\,\mathrm{fin.}\,(\varphi - \pi) + T\,\mathrm{cof.}\,(\varphi - \pi) - \frac{R}{\mathrm{tang.}\,G}\right).$$

Unde latitudo ψ ita definitur ut ſit $\mathrm{tang.}\,\psi = \mathrm{fin.}\,(\varphi - \pi)\,\mathrm{tang.}\,G$. Hic quidem partes adſunt à viribus perturbatricibus non pendentes, verum hoc inde venit, quod vim quadratis diſtantiarum reciproce proportionalem in plano fixo aſſumſimus. Si enim ea, uti rei natura poſtulat, ſecundum diſtantiam veram RC aſſumatur, iſtæ partes à vi $\frac{R}{\mathrm{tang.}\,G}$ tollentur, id quod in applicatione fiet manifeſtum.

SECTIO III.

SECTIO III.

Inveſtigatio Virium quibus motus Planetæ principalis ab actione alius Planetæ perturbatur.

§. XXXVII. CUM perturbationes, quibus planetæ principales ſe mutuo afficiunt, ſint vehementer parvæ, dum uniuſcujuſque planetæ perturbationes inveſtigamus, motum reliquorum tanquam regulis Kepleri perfecte conformem ſpectare licebit: tantillus enim error, qui hac ratione in motu planetæ perturbantis admittitur, in effectu multo minorem, hoc eſt evaneſcentem producere eſt cenſendus. Quoniam igitur quæſtio ad planetas principales adſtringitur, punctum fixum C in centro ſolis aſſumi conveniet; & quia motus planetæ perturbantis in plano fieri poteſt judicari, hoc ipſum planum pro plano illo fixo, ad quod motum planetæ turbati referre conſtituimus, commodiſſime aſſumemus. Cum enim invenerimus, quomodo motus iſtius planetæ reſpectu hujus plani immutetur, facile erit perturbationes ad quodvis aliud planum in cœlo fixum traducere, ſicque inconſtantiam, cui planum orbitæ planetæ perturbantis eſt obnoxium, exuere.

§. XXXVIII. Conveniat igitur planum tabulæ cum plano orbitæ planetæ perturbantis, in quo C ſit centrum ſolis, & CA recta inde ad fixum cœli punctum ducta, unde longitudines numerentur. Ad datum ergo tempus t planeta perturbans ſit in hujus plani puncto V, à quo ductâ rectâ CV, ponatur: *FIG. II.*

I. Distantia hujus planetæ à sole . . . $CV = y$

II. Longitudo ejus seu angulus ACV . . . $= \theta$

Quod si ergo vis, qua hic planeta in V ad solem urgetur sit $= \frac{ff}{yy}$, ejusque orbitæ semi latus rectum, seu semi-parameter $= c$, ejus excentricitas $= e$, & anomalia vera $= u$, erit per formulas supra inventas:

$$y = \frac{c}{1 - e\,cos.\,u};\ \&\ d\theta = du = \frac{f\,dt}{yy}\sqrt{\tfrac{1}{2}c} = \frac{f\,dt\,(1 - e\,cos.\,u)^2}{c\sqrt{2c}};$$

unde fit $dy = -\frac{c\,e\,du\,sin.\,u}{(1 - e\,cos.\,u)^2} = -\frac{e\,f\,dt\,sin.\,u}{\sqrt{2c}}$; sicque hæc differentialia ad elementum temporis dt habemus reducta.

§. XXXIX. Alter jam planeta, cujus perturbationes motus indagamus, extra hoc planum reperiatur in R unde ad id ducto perpendiculo RQ, junctisque rectis RC & QC, sit ut ante:

I. Ejus distantia à sole curtata seu recta $CQ = x$

II. Longitudo ejus seu angulus ACQ . . $= \varphi$

III. Latitudo ejus seu angulus QCR . . . $= \psi$

Unde fit ejus distantia à sole vera . . . $= \frac{x}{cos.\,\psi}$

Verum pro latitudine considerentur linea nodorum $C\Omega$ & inclinatio orbitæ ad planum orbitæ prioris planetæ, sitque

IV. Longitudo nodi ascendentis seu angulus $AC\Omega = \pi$

V. Inclinatio orbitæ seu angulus QSR . . . $= G$

ex quibus elementis latitudo, ita exprimitur, ut sit

$$tang.\,\psi = sin.\,(\varphi - \pi)\ tang.\,G.$$

Quod si porro jungantur rectæ QV & RV, erit

$$QV = \sqrt{(xx + yy - 2xy\,cos.\,(\varphi - \theta))},\ \&$$

$$VR = \sqrt{\left(\frac{xx}{cos.\,\psi^2} + yy - 2xy\,cos.\,(\varphi - \theta)\right)} = z$$

ponamus enim brevitatis gratia hanc distantiam $VR = z$.

§. XL. Cum jam planeta R tam ad solem quam ad alterum planetam V attrahatur in ratione reciproca duplicata distantiarum, sit utraque vis ita comparata, ut ad distantiam d habeatur

vis acceleratrix ad solem tendens . . . $= \frac{E}{dd}$,

vis acceleratrix ad planetam tendens . . . $= \frac{F}{dd}$.

Hinc itaque planeta in R urgebitur

primo ad solem secundum RC vi $= \frac{E\,cos.\,\psi^2}{xx}$;

deinde ad planetam in V secundum RV vi $= \frac{F}{zz}$.

Quoniam vero ipse sol quoque ad planetam V sollicitatur secundum CV vi acceleratrice $= \frac{F}{yy}$, ut solem in quiete retineamus hæc vis secundum directionem contrariam $Q\nu$ in planetam R transferri debet, hincque iste præterea sollicitabitur:

Secundum directionem $Q\nu$ vi $= \frac{F}{yy}$.

Quin etiam ipse planeta V omnino ad solem urgeri censendus est secundum VC vi $= \frac{E+F}{yy}$, quam modo posueramus $= \frac{ff}{yy}$.

§. XLI. Nunc vero ante omnia vires planetam R sollicitantes revocari debent ad directiones QC, QN & RQ, ut inde valores virium assumtarum V, T & R obtineantur. Ac primo quidem vis secundum $RC = \frac{E\,cos.\,\psi^2}{xx}$ præbet

Secundum directionem QC vim $= \frac{E\,cos.\,\psi^3}{xx}$ pro vi V;

Secundum directionem RQ vim $= \frac{E\,sin.\,\psi\,cos.\,\psi^2}{xx}$ pro vi R;

Secunda vis secundum $RV = \frac{F}{zz}$, ob $QR = x\, tang.\, \psi$, & $VR = z$, præbet

Secundum directionem RQ vim $= \frac{F.QR}{z^3} = \frac{Fx\, tang.\, \psi}{z^3}$ pro R,

Tum vero secundum QV vim $= \frac{F.QV}{z^3}$ quæ, ob

$$QV\, cos.\, CQV = -y\, cos.\, (\varphi - \theta) + x \;\&$$
$$QV\, sin.\, CQV = y\, sin.\, (\varphi - \theta),$$

reducitur ad binas sequentes

Secundum QC vim $= \frac{-F}{z^3}(y\, cos.\, (\varphi - \theta) - x)$ pro vi V,

Secundum QN vim $= \frac{Fy\, sin.\, (\varphi - \theta)}{z^3}$ pro vi T;

Denique tertia vis secundum $Q\nu = \frac{F}{yy}$, ob $CQ\nu = \varphi - \theta$, dat

Secundum QC vim $= \frac{F\, cos.\, (\varphi - \theta)}{yy}$ pro vi V,

Secundum QN vim $= \frac{-F\, sin.\, (\varphi - \theta)}{yy}$ pro vi T.

§. XLII. Colligamus singulas has vires ad directiones QC, QN & RQ reductas, atque habebimus

Vim $V = \frac{E\, cos.\, \psi^2}{xx} + \frac{Fx}{z^3} - Fy\,(\frac{1}{z^3} - \frac{1}{y^3})\, cos.\, (\varphi - \theta)$,

Vim $T = Fy\,(\frac{1}{z^3} - \frac{1}{y^3})\, sin.\, (\varphi - \theta)$,

Vim $R = \frac{E\, sin.\, \psi\, cos.\, \psi^2}{xx} + \frac{Fx\, tang.\, \psi}{z^3}$;

nihilque superest, nisi ut hæ expressiones in locum litterarum V, T & R substituantur. Quoniam vero angulus ψ semper est valde parvus, ejus cosinus proxime ad unitatem accedet; unde cum posuerimus $V = \frac{ff}{xx} + S$, ita ut sit

$$S = \frac{-ff}{xx} + \frac{E\, cos.\, \psi^2}{xx} + \frac{Fx}{z^3} - Fy\,(\tfrac{1}{z^3} - \tfrac{1}{y^3})\, cos.\, (\varphi - \theta),$$

pro ff assumi poterit E; & quamquam ante inveneramus $ff = E + F$, tamen quantitas F præ E tam est exigua, ut nullus plane error sit metuendus, si ponamus $ff = E$; ita ut habeamus:

$$S = \frac{-E(1 - cos.\, \psi^2)}{xx} + \frac{Fx}{z^3} - Fy\,(\tfrac{1}{z^3} - \tfrac{1}{y^3})\, cos.\, (\varphi - \theta).$$

§. XLIII. Antequam autem hujus reductionis rationem habeamus, substitutio virium V, T & R in æquationibus variationem lineæ nodorum & inclinationis continentibus commode fieri poterit, quæ eo magis est notatu digna, quod uti jam innuimus termini à vi perturbatrice non pendentes destruantur. Cum enim sit $tang.\, G = \frac{tang.\, \psi}{sin.\, (\varphi - \pi)}$ factoris $V\, sin.\, (\varphi - \pi) + T\, cos.\, (\varphi - \pi) - \frac{R}{tang.\, G.}$, qui illas expressiones ingreditur, valor satis concinne definietur, habebitur namque:

$$V\, sin.\, (\varphi - \pi) = \frac{E\, cos.\, \psi^2}{xx}\, sin.\, (\varphi - \pi) + \frac{Fx}{z^3}\, sin.\, (\varphi - \pi) - Fy\,(\tfrac{1}{z^3} - \tfrac{1}{y^3})\, cos.\, (\varphi - \theta)\, sin.\, (\varphi - \pi),$$

$$T\, cos.\, (\varphi - \pi) = Fy\,(\tfrac{1}{z^3} - \tfrac{1}{y^3})\, sin.\, (\varphi - \theta)\, cos.\, (\varphi - \pi),$$

$$-\frac{R}{tang.\, G.} = \frac{-E\, cos.\, \psi^2}{xx}\, sin.\, (\varphi - \pi) - \frac{Fx}{z^3}\, sin.\, (\varphi - \pi);$$

Quare cum sit $-cos.\, (\varphi - \theta)\, sin.\, (\varphi - \pi) + sin.\, (\varphi - \theta)\, cos.\, (\varphi - \pi) = -sin.\, (\theta - \pi)$, erit $V\, sin.\, (\varphi - \pi) + T\, cos.\, (\varphi - \pi) - \frac{R}{tang.\, G} = -Fy\,(\frac{1}{z^3} - \frac{1}{y^3})\, sin.\, (\theta - \pi)$.

§ XLIV. Perspicuum ergo est totam hanc expressionem, qua variatio in linea nodorum & inclinatione determinatur, unice à vi perturbante F pendere; cæteriſque paribus ſinui anguli $\theta - \pi$, qui oritur longitudinem nodi $\pi\,a$ longitudine planetæ turbantis θ ſubtrahendo, eſſe proportionalem. Quodſi ergo ut ante ſemiparametrum orbitæ planetæ R ponamus $= p$, ex §. XXXVI, pro variatione tam lineæ nodorum, quam inclinationis, ſequentes nanciſcemur æquationes:

$$d\pi = -\frac{Fxy\,dt\,\text{ſin.}(\varphi - \pi)\,\text{ſin.}(\theta - \pi)}{f\sqrt{2p}}\left(\frac{1}{z^3} - \frac{1}{y^3}\right),$$

$$d.\,l.\,tang.\,G = -\frac{Fxy\,dt\,\text{coſ.}(\varphi - \pi)\,\text{ſin.}(\theta - \pi)}{f\sqrt{2p}}\left(\frac{1}{z^3} - \frac{1}{y^3}\right);$$

Ubi imprimis eſt memorabile has expreſſiones tanto ſimpliciores prodiiſſe. Quamvis autem eædem facilius ex ipſa virium ſollicitantium indole erui potuiſſent, tamen earum derivationem ex formulis generalibus petere convenientius eſt viſum.

§. XLV. Progrediamur ergo ad reliquas perturbationes, & cum poſuerimus pro orbita planetæ R turbata:

Semiparametrum . . . . $= p$,
Excentricitatem . . . . $= q$,
Et anomaliam veram . . . . . $= \nu$; ita ut ſit

$$x = \frac{p}{1 - q\,\text{coſ.}\,\nu} \;\&\; dx = \frac{-fq\,dt\,\text{ſin.}\,\nu}{\sqrt{2p}},$$

primo variationem parametri ita invenimus expreſſam

$$dp = -\frac{Tx\,dt}{f}\sqrt{2p} = -\frac{Tp\,dt\sqrt{2p}}{f(1 - q\,\text{coſ.}\,\nu)},$$

quæ ideoque hanc induet formam:

$$dp = -\frac{Fxy\,dt\,\text{ſin.}(\varphi - \theta)}{f}\left(\frac{1}{z^3} - \frac{1}{y^3}\right)\sqrt{2p},$$

neque enim adhuc pro x, y & z valores ſupra deſignatos ſubſtitui conveniet, quia illi nonſolum jam ſunt

cogniti, ſed etiam eorum differentialia per dt exhiberi poſſunt

ob $y = \dfrac{c}{1 - e\,\text{coſ.}\,u}$, $d\theta = du = \dfrac{f\,dt}{yy}\sqrt{\tfrac{1}{2}c}$, &

$$dy = -\frac{fe\,dt\,\text{ſin.}\,u}{\sqrt{2c}}.$$

§. XLVI. Secundo excentricitatis q variatio §. XXXIII eſt inventa

$$dq = \frac{dt}{f}\left(T\,\text{coſ.}\,\nu + S\,\text{ſin.}\,\nu + T.\frac{\text{coſ.}\,\nu - q}{1 - q\,\text{coſ.}\,\nu}\right)\sqrt{\tfrac{1}{2}p}, \text{ ſeu}$$

$$dq = \frac{dt}{f}\left(2T\,\text{coſ.}\,\nu + S\,\text{ſin.}\,\nu - \frac{Tq\,\text{ſin.}\,\nu^2}{1 - q\,\text{coſ.}\,\nu}\right)\sqrt{\tfrac{1}{2}p},$$

ubi recordari debemus eſſe:

$$S = \frac{-E(1 - \text{coſ.}\,\psi^2)}{xx} + \frac{Fx}{z^3} - Fy\left(\frac{1}{z^3} - \frac{1}{y^3}\right)\text{coſ.}(\varphi - \theta), \;\&$$

$$T = Fy\left(\frac{1}{z^3} - \frac{1}{y^3}\right)\text{ſin.}(\varphi - \theta);$$

quorum valorum ſubſtitutionem fieri non eſt opus: quod cum etiam in reliquis commode fieri nequeat, eas apponamus uti invenimus §. XXXV. Tertio ſcilicet pro motu lineæ abſidum obtinuimus

$$d\varphi - d\nu = \frac{dt}{fq}\left(2T\,\text{ſin.}\,\nu - S\,\text{coſ.}\,\nu + \frac{Tq\,\text{ſin.}\,\nu\,\text{coſ.}\,\nu}{1 - q\,\text{coſ.}\,\nu}\right)\sqrt{\tfrac{1}{2}p};$$

& quia $d\varphi = \dfrac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p} = \dfrac{f\,dt(1 - q\,\text{coſ.}\,\nu)^2}{p.\sqrt{2p}}$,

erit incrementum momentaneum anomaliæ veræ

$$d\nu = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p} - \frac{dt}{fq}\left(2T\,\text{ſin.}\,\nu - S\,\text{coſ.}\,\nu + \frac{Tq\,\text{ſin.}\,\nu\,\text{coſ.}\,\nu}{1 - q\,\text{coſ.}\,\nu}\right)\sqrt{\tfrac{1}{2}p}.$$

§. XLVII. Nunc autem elementum temporis dt eliminari conveniet, cujus loco commodiſſime motus medius ſolis ſive terræ introducitur. Ponamus ergo diſtantiam mediam terræ à ſole eſſe $= a$, hocque motu medio abſolvi tempuſculo dt angulum $= d\omega$.

Quoniam hoc caſu excentricitas adeſt nulla, viſque ad ſolem tendens eſt $=\frac{ff}{aa}=\frac{E}{aa}$, erit ex principiis ante ſtabilitis: $d\omega=\frac{fdt}{aa}\sqrt{\frac{1}{2}a}=\frac{fdt}{a\sqrt{2a}}$ ideoque $f\,dt=a\,d\omega\sqrt{2a}$, &

$$\frac{dt}{F}=\frac{a\,d\omega\sqrt{2a}}{ff}=\frac{a\,d\omega\sqrt{2a}}{E}.$$

Tempore ergo abſoluto t eliminato, ejuſque loco angulo ω quem ſol motu medio interea abſolvit, noſtræ formulæ differentiales omnes ad elementum iſtius motus medii $d\omega$ reduci poterunt. Ac ſi inſuper ponatur $F=nE$, ubi n ſemper fractionem vehementer parvam denotabit, erit $\frac{T}{E}=ny\left(\frac{1}{z^3}-\frac{1}{y^3}\right)\mathit{ſin.}\,(\varphi-\theta)$,

$$\frac{S}{E}=-\frac{(1-\mathit{coſ.}\,\varphi^2)}{xz}+\frac{nx}{z^3}-ny\left(\frac{1}{z^3}-\frac{1}{y^3}\right)\mathit{coſ.}\,(\varphi-\theta).$$

§. XLVIII. Subſtituto ergo pro dt iſto valore, habebimus primo pro planeta perturbante:

$d\theta=du=\frac{a\,d\omega}{yy}\sqrt{ac}$; & $dy=-a\,e\,d\omega\,\mathit{ſin.}\,u\sqrt{\frac{a}{c}}$,

at pro planeta perturbato:

$d\varphi=\frac{a\,d\omega}{xx}\sqrt{ap}$; & $dx=-a\,q\,d\omega\,\mathit{ſin.}\,v\sqrt{\frac{a}{p}}$ porroque

$$dp=-\frac{T}{E}.\,2ax\,d\omega\sqrt{ap}=-2naxy\,d\omega\left(\frac{1}{z^3}-\frac{1}{y^3}\right)\mathit{ſin.}(\varphi-\theta).\sqrt{ap},$$

$$dq=a\,d\omega\left(\frac{2T}{E}\mathit{coſ.}\,v+\frac{S}{E}\mathit{ſin.}\,v-\frac{T}{E}.\frac{q\,\mathit{ſin.}\,v^2}{1-q\,\mathit{coſ.}\,v}\right).\sqrt{ap},$$

$$d\varphi-dv=\frac{a\,d\omega}{q}\left(\frac{2T}{E}\mathit{ſin.}\,v-\frac{S}{E}\mathit{coſ.}\,v+\frac{T}{E}.\frac{q\,\mathit{ſin.}\,v\,\mathit{coſ.}\,v}{1-q\,\mathit{coſ.}\,v}\right).\sqrt{ap},\ \&$$

$$dv=\frac{a\,d\omega}{xx}.\sqrt{ap}-\frac{a\,d\omega}{q}\left(\frac{2T}{E}\mathit{ſin.}\,v-\frac{S}{E}\mathit{coſ.}\,v+\frac{T}{E}.\frac{q\,\mathit{ſin.}\,v.\,\mathit{coſ.}\,v}{1-q\,\mathit{coſ.}\,v}\right).\sqrt{ap}.$$

Si ulterius ſemi-axis tranſverſus $\frac{p}{1-qq}$ ponatur $=r$, erit

$$dr=\frac{-2arr\,d\omega\sqrt{a}}{\sqrt{p}}\left(\frac{T}{E}-\frac{T}{E}q\,\mathit{coſ.}\,v-\frac{S}{E}q\,\mathit{ſin.}\,v\right),$$

uti ex §. XXXIV colligere licet,

§. XLIX.

§. XLIX. Simili modo & mutationes, quas linea nodorum & inclinatio orbitæ tempuſculo, quo ſol ſecundum medium motum per angulum $d\omega$ progreditur ſubeunt, exprimi poterunt. Poſito enim $F=nE$, & ſolis diſtantia media à terra $=a$, quoniam invenimus:

$$\frac{dt}{F}=\frac{na\,d\omega\sqrt{2a}}{E}\ \text{erit}\ \frac{Fdt}{f}=na\,d\omega\sqrt{2a},\ \&\ \frac{Fdt}{f\sqrt{2p}}=na\,d\omega\sqrt{\frac{a}{p}};$$

formulæ ſupra exhibitæ abibunt in has:

$$d\pi=-naxy\,d\omega\,\mathit{coſ.}(\varphi-\pi)\,\mathit{ſin.}(\theta-\pi).\left(\frac{1}{z^3}-\frac{1}{y^3}\right).\sqrt{\frac{a}{p}}.$$

$$d.\,l.\,tang.\,G=-naxy\,d\omega\,\mathit{coſ.}(\varphi-\pi)\,\mathit{ſin.}(\theta-\pi).\left(\frac{1}{z^3}-\frac{1}{y^3}\right).\sqrt{\frac{a}{p}}.$$

Cum autem valor ipſius $d\pi$ jam fuerit inventus, erit ſuccinctius

$$d.\,l.\,tang.\,G=\frac{d.\,tang.\,G}{tang.\,G}=d\pi\frac{\mathit{coſ.}\,(\varphi-\pi)}{\mathit{ſin.}\,(\varphi-\pi)}.$$

Per has igitur formulas omnium quantitatum, quibus determinatio perturbationum continetur, incrementa, quæ tempuſculo per motum ſolis medium $d\omega$ expreſſo capiunt, definiri poterunt, neque hactenus ulla approximatione ſumus uſi, niſi quatenus pro motu planetæ perturbantis loco $E+F$ ſimpliciter E ſcripſimus, unde autem nulla aberratio à vero oriri poteſt.

§. L. At vero in calculi ſubſidium jam multo graviorem hypotheſin aſſumſimus, dum motum planetæ perturbantis, quem in V fingimus, tanquam regulis Kepleri perfecte conſentaneum ſpectamus; ſi enim iſte planeta eſſet Saturnus, cujus motum non mediocriter ab actione Jovis turbari novimus, nullum certè eſt dubium, quin ejus perturbationes effectus, qui ab ejus actione in motum reliquorum planetarum redundant, aliquantillum eſſent affecturæ. Interim tamen pro certo ſtatuere licet, iſtas variationes incomparabiliter futuras eſſe minores, neque effectum Saturni verum ſenſibiliter eſſe diſcrepaturum ab eo, quem regulas Keppleri exacte ſecutus, eſſet producturus; imprimis cum conſtet, univerſam perturbationem eſſe quam minimam, atque adeo nos contenti eſſe debeamus eam tantum vero proxime determinaſſe. Æque parvi autem momenti ſine ullo dubio æſtimanda erit ea aberratio, quæ dum pro $E + F$ tantum E ſeu 1 pro $1 + n$ ſemper eſt fractio quam minima.

§. LI. Non parum paradoxon videri debet, quod etiamſi vis planetæ perturbantis, ſeu fractio n penitus evaneſceret, tamen pro altero planeta tam excentricitas q quam linea abſidum mutationibus eſſet obnoxia: propterea quod evaneſcente fractione n, quantitas $\frac{S}{E}$ non in nihilum abeat, ſed valorem $= \frac{-1 + cos. \psi^3}{xx}$ retineat, quo utrumque differentiale dq & $d\varphi - dv$ afficitur. Verum perpendendum eſt, quod dum orbitam planetæ in aliud planum projicimus, projectio quidem quoque futura ſit ellipſis, ſed cujus focus non amplius futurus ſit in puncto C. Etiamſi ergo motus projectus fiat in ellipſi, areæque adeo circa punctum C deſcriptæ temporibus ſint proportionales, tamen quia in C non eſt focus ellipſis, motus regulis Keppleri non erit conformis. Cum autem nihilominus fingi poſſet quovis momento ellipſis focum habens in C, cujus elementum cum illius ellipſis elemento congruat, mirum non eſt hujus ellipſis fictæ tam excentricitatem quam poſitionem lineæ abſidum continuo variari: notatu autem eſt dignum parametrum ellipſis ſemper invariatum relinqui.

§. LII. Cum igitur caſu $n = o$ hoc incommodum penitus evitaremus, ſi motum planetæ non in plano alieno ſed proprio contemplaremur, idem quoque incommodum in genere evitabimus, ſi motum quovis

tempore ad planum orbitæ ipſum referamus, ita ut x diſtantiam veram CR & φ longitudinem planetæ in proprio plano denotaret. Hinc quidem ob orbitæ inconſtantiam alia incommoda naſcerentur, quæ autem, dummodo inclinatio G ſit valde parva, quemadmodum id quidem ſemper uſu venit, fere penitus removebuntur, propterea quod inde ipſis perturbationibus minima perturbatio inducetur. Quo circa totum negotium ita promtius expedietur ut primo dum variationes quantitatum p, q, & $\varphi - v$, exquiruntur, à quibus locus planetæ in propria orbita pendet, latitudo ψ prorſus negligatur ponendo $\psi = o$; deinde vero ſeorſim ad quodvis tempus tam poſitio lineæ nodorum quam inclinatio inveſtigetur: & his denique inventis more apud Aſtronomos recepto ex loco planetæ in propria orbita ejuſque argumento latitudinis ipſa latitudo eliciatur, cum reductione longitudinis.

§. LIII. Hanc ob cauſam quæſtionem noſtram commode bipartito pertractare licebit dum ſeorſim primo motus planetæ in propria orbita, quaſi eſſet plana, deinde vero hujus orbitæ poſitio reſpectu orbitæ planetæ perturbantis inveſtigatur. Primo igitur x denotabit diſtantiam veram planetæ à ſole, & φ ejus longitudinem; tum poſito:

$z = \sqrt{(xx + yy - 2xy\, cos.\, (\varphi - \theta))}$, ob $\psi = o$,

ſi brevitatis gratia ſtatuamus

$M = y\left(\frac{1}{z^3} - \frac{1}{y^3}\right) ſin.\, (\varphi - \theta)$,

$N = \frac{x}{z^3} - y\left(\frac{1}{z^3} - \frac{1}{y^3}\right) cos.\, (\varphi - \theta)$; habebimus:

$$y = \frac{c}{1 - e\, cos.\, u};\ dy = -\, a e\, d\omega\, ſin.\, u. \sqrt{\frac{a}{c}};$$

$$d\theta = du = \frac{a\, du}{yy}\sqrt{ac};$$

$$x = \frac{p}{1 - q\, cos.\, v};\ dx = -\, a q\, d\omega\, ſin.\, v. \sqrt{\frac{a}{p}};$$

$d\varphi = \frac{a\,d\omega}{xx} \sqrt{ap}$; itemque:

$dp = -2nMax\,d\omega . \sqrt{ap}$:

$dq = na\,d\omega \left(2M\,cos.\,v + N\,sin.\,v - \frac{Mq\,sin.\,v^2}{1 - q\,cos.\,v}\right)\sqrt{ap}$, &

$d\varphi - dv = \frac{na\,d\omega}{q}\left(2M\,sin.\,v - N\,cos.\,v + \frac{Mq\,sin.\,v\,cos.\,v}{1 - q\,cos.\,v}\right)\sqrt{ap}$;

ubi est $n = \frac{F}{E}$, cujus valor pro singulis planetis ex observationibus, quantum quidem id fieri licet, concludi debet.

§. LIV. Cum deinceps ex his formulis locus planetæ R in propria orbita cum ejus distantia à sole fuerit inventus, porro investigetur ad quodvis tempus propositum positio hujus orbitæ respectu plani fixi assumti, in quo orbita planetæ perturbantis versatur, linea scilicet nodorum cum inclinatione mutua, ope formularum in §. XLIX exhibitarum; hincque facillime verus locus planetæ in cœlo assignabitur. In priori quidem investigatione neglectio partis $\frac{-1 + cos.\,\psi^2}{xx}$ in valore N nullum errorem creat, quippe quæ per reductionem ad propriam orbitam compensatur. Sed ob positionem $\psi = o$, valor ipsius z aliquantillum immutatur, sed tam parum, nisi inclinatio sit enormis, ut error præ ipsa quantitate z sit vehementer exiguus. Quoniam igitur ipsa quantitas z aliter non intrat in calculum nisi per fractionem minimam n multiplicat errores illi denuo hinc diminuentur, ut tuto pro nihilo æstimari queant.

SECTIO IV.

Considerationes necessariæ ad resolutionem formularum inventarum expediendam.

§. LV. Ex his formulis primo sine integrationis adjumento variationes horariæ, quas singula motus elementa capiunt intervallo unius horæ, satis exacte colligi possunt, cum enim tantillo tempore omnes variationes sint quam minimæ, error plane erit imperceptibilis si ipsa differentialia tanquam variationes horarias spectemus. Cum igitur Sol secundum motum medium una hora angulum conficiat $= 2'\,28''$ seu accuratius $147\frac{5}{6}''$, si differentiali $d\omega$ hunc valorem tribuamus, ut sit $d\omega = 147\frac{5}{6}''$, seu in partibus radii, pro quo unitatem assumimus, $d\omega = 0{,}00071672$, reliqua differentialia dp, dq, $d\varphi - dv$, $d\varphi$, dx, $d\pi$, $d.\,tang.\,G$, $d\theta = du$, & dy incrementa horaria istorum elementorum exhibebunt, quæ igitur sine ulla integratione definire licebit, dummodo pro tempore proposito ipsa hæc elementa fuerint cognita. Neque etiam error erit sensibilis, si hoc modo variationes diurnas tribuendo ipsi $d\omega$ valorem vicies quater majorem definire vellemus, dummodo neutrius planetæ motus tempore unius diei admodum sit notabilis.

§. LVI. Si quis hunc laborem suscipere vellet, totum negotium sine integratione expedire posset. Cum enim pro dato quopiam tempore t explorati fuerint valores elementorum singulorum, quibus determinatio motus continetur, inventis singulorum incrementis horariis, colligemus hinc eorundem elementorum valores ad tempus $t+1$ hora; unde deinceps simili modo eadem elementa ad tempus $t+2$ horis obtinebimus, sicque ad tempora quotcunque horis remota progredi licebit. Interim tamen quia in singulis gradibus error quidam, etiamsi in se spectatus sit insensibilis committitur, is per continuam repetitionem ita accumulabitur, ut tandem satis notabilis evadat. Neque etiam is minoribus hora intervallis assumendis quo pacto quidem labor omnino insuperabilis redderetur, evitari posset, etsi enim singuli errores fierent multo minores tamen ob majorem operationum numerum tandem quoque ad magnitudinem notabilem excrescere possent.

§. LVII. Nisi igitur integratio in subsidium vocetur, sperari omnino nequit, ut singulas motus perturbationes unquam exacte definire valeamus, vel saltem ut tempore quantum vis magno interjecto error non fiat notabilis. Huc quoque accedet, quod priori methodo utentes ad nullum tempus, unde calculum inchoare vellemus, per observationes singulorum elementorum veros valores assignare valeremus, ideoque etiam hi errores sequentes operationes plurimum contaminarent. Quando autem integrationes in genere perficere licuerit, tum per observationes plurimas diversis temporibus institutas, dum eæ cum calculo generali conferentur, veri singulorum elementorum valores colligi poterunt, quibus semel definitis formulæ integratæ perpetuo usum desideratum præstabunt. Quocirca ad perfectam omnium perturbationum cognitionem absolute necessarium est, ut formulæ differentiales inventæ per integrationem ad determinationes finitas reducantur.

§. LVIII. Ac per reductiones quidem hactenus factas jam eximium commodum sumus consecuti, quod formulas differentio-differentiales, ad quas principia Mechanica nos immediate perduxerant, ad formulas simpliciter differentiales revocaverimus, quæ nullis amplius

formulis integralibus sint involutæ, quemadmodum usu venit in iis quas primum (§. XXI) elicueramus, quæ partim ob irrationalitatem partim ob integrales, vix tractari potuissent. Præcipuum autem commodum sine dubio in hoc consistere est censendum, quod omnes formulas ad similitudinem motus regularis, seu notissimis Keppleri regulis conformis explicuerimus, quæ reductio quoque ad usum Astronomicum maxime videtur accommodata. Neque etiam amplius premimur ejusmodi formulis irrationalibus, quarum valores ita sint vagi, ut modo in nihilum abire, modo etiam negativi fieri queant, uti in formulis §. XXI usu venerat.

§. LIX. Verum antequam integrationis negotium suscipiamus, nonnulla moneri est necesse, quibus operationes instituendæ dirigantur. Cum enim integrationem absolutam ac perfectam nullo modo sperare queamus, ad approximationes confugere cogimur, in quo negotio cum multa arbitrio nostro relinquantur, prouti alias atque alias particulas negligere velimus, præcipua cura hoc erit collocanda, ut nihil negligamus, unde error sensibilis resultare posset. Ex circumstantiis igitur judicari opportebit, quid ratione singulorum elementorum negligere liceat: ac primo quidem cum n sit fractio tantopere exigua, quippe qua ratio massæ planetæ perturbantis ad massam solis exprimitur, nullum est dubium, quin ejusmodi terminos, qui per quadratum hujus fractionis altioremve potestatem essent multiplicati, sine hæsitatione rejicere queamus. Ex ipsa autem hujus numeri n parvitate cognoscimus, perturbationes esse quam minimas, quæ adeo omnes evanescerent si esset $n=o$.

§. LX. Deinde etiam si orbitas singulorum planetarum consideremus, earum excentricitates tam parvas deprehendimus, ut in determinatione perturbationum, si non ipsæ, tamen earum quadrata altioresque potestates tuto negligi possint. De Mercurio quidem & Marte hic dubium suboriri posset, quorum planetarum excentricitas est maxima, è contrario eorum massæ tam sunt exiguæ præ massa Solis, ut totæ perturbationes inde oriundæ fere contemni queant. Quamquam autem in reliquorum planetarum motu proprio determinando quadratum excentricitatis perperam negligeretur, unde plerumque effectus satis sensibilis oriri solet, tamensi perturbationes quæ ab ejus actione in alios planetas redundant, investigentur, quoniam effectus quadrati insuper per fractionem n multiplicatur, is plane omnino imperceptibilis reddetur. Hinc quantitates e & q hujusque quantitatem mediam k tamquam tam parvas assumemus, ut in perturbationum investigatione earum quadrata certe ac plerumque etiam eæ ipsæ tuto rejici queant, quod dum evolutionem instituemus, clarius perspicietur.

§. LXI. Neque tamen pro planeta perturbato excentricitas q nimis parva concipi potest sed saltem tam magna, ut mutationes quas ipsi actio reliquorum planetarum inducere valet, præ tota ejus magnitudine tanquam minimæ spectari queant. Cum enim formula progressionem aphelii exprimens divisa sit per q, evidens est si valor ipsius q nimis esset exiguus, ac fortasse interdum plane evanesceret, motum aphelii maximis difficultatibus impeditum iri, ita ut si talis casus in mundo existeret, vix quicquam ex nostris formulis concludi liceret. Verum hic iterum ad insigne nostrum commodum usu venit, ut nullius planetæ excentricitas tam sit exigua, ut inde quicquam nobis sit extrimescendum. Cum enim Veneris excentricitas sit omnium minima, nullum tamen est dubium, quin mutationes, quas ea unquam ab actione reliquorum planetarum subit, præ ejus valore medio quasi evanescant. Quare si valor medius excentricitatis q ponatur $= k$, differentia inter k
&

& q præ k seu fractio $\frac{k-q}{k}$ semper tanquam minima tuto spectari poterit.

§. LXII. Maximam autem in hac investigatione molestiam nobis facessit valor quantitatis $z = \sqrt{(xx + yy - 2xy. cos. (\phi - \theta))}$ qui eo magis est variabilis, quo propius amborum planetarum orbitæ ad se invicem accedunt. In calculo quidem hic valor, uti est irrationalis, relinqui non potest, quia integrationes instituendæ nullo modo succederent. Cum enim calculum aliter tractare non liceat, nisi ut omnes perturbationes ad sinus cosinusve certorum angulorum revocentur, omnino necesse est, ut quantitates M & N, quæ hanc quantitatem z involvunt, in ejusmodi series evolvant z, quæ hujusmodi sinus vel cosinus simpliciter contineant. Hoc enim solo modo integratio suscipi posse videtur, neque etiam ulla via patet, quemadmodum calculus ita expediri possit, ut valores integrales adhuc quantitatem z aliasve independentes in se contemplectentur.

§. LXIII. Insignem hic Analyseos, quatenus quidem etiamnunc est exculta, defectum agnoscere cogimur, quod aliter formularum inventarum integralia exhibere non valeamus, nisi per series, quarum singuli termini simplices sinus vel cosinus angulorum $\phi - \theta$, u, v, $\phi - \pi$, $\theta - \pi$, ex iisque compositorum contineant. Fieri certe posset ut vera integralia vel quantitates ex his complexas neque in hujusmodi serie commode resolubiles continerent, vel etiam alios angulos veluti CVQ vel CQV qui utique si vis principalis ad planetam V tenderet, numerusque n foret prægrandis, primarias partes in calculo essent obtenturi. Atque cum ab his angulis, si n esset numerus valde magnus, totus calculus maximam partem penderet, eo minus dubitare possumus, quin iidem etiam præsenti casu, si calculum accurate expedire liceret, sint ingressuri. Interim tamen plane non patet, quomodo isti anguli per integrationem in calculum invehi queant. Quod si ergo ex hac parte fines analyseos extendere unquam contigerit, tum demum majores fructus pro Astronomia nobis polliceri poterimus.

§. LXIV. Eo igitur sumus redacti ut quantitatem surdam $z = \sqrt{(xx + yy - 2xy \cos.(\varphi - \theta))}$ in seriem transformemus, ubi quidem eo potissimum est incumbendum, ut ista series quantum fieri potest reddatur convergens, atque secundum sinus vel cosinus certorum angulorum progrediatur. Et quoniam usus postulat convergentiam sive x sit majus, sive minus quam y, terminus tantum $2xy \cos.(\varphi - \theta)$, qui modo affirmativus, modo in nihilum abire, modo negativus fieri potest, molestiam creat. Binomii ergo partem alteram constituo $xx + yy$, alteram vero $2xy \cos.(\varphi - \theta)$; statuoque:

$$xx + yy = rr, \ \& \ \frac{2xy}{xx+yy} = s; \ \text{ut sit}$$

$$z = \sqrt{(rr - rrs \cos.(\varphi - \theta))} = r\sqrt{(1 - s \cos.(\varphi - \theta))},$$

atque hic perspicuum est s semper esse unitate minus nisi sit $x = y$, & eo fieri minus, quo magis distantiæ x & y fuerint inter se inæquales: hinc ergo multo magis pars $s \cos.(\varphi - \theta)$ minor erit parte 1, prout seriei convergentia postulat.

§. LXV. Quoniam angulus $\varphi - \theta$ tum frequenter occurret, ponamus ad abbreviandum $\varphi - \theta = \eta$, & cum sit $r = \sqrt{(xx + yy)}$ & $s = \frac{2xy}{xx+yy}$, erit $z = r\sqrt{(1 - s \cos.\eta)}$, ideoque $\frac{1}{z^3} = \frac{1}{r^3}(1 - s \cos.\eta)^{-\frac{3}{2}}$; unde formulam irrationalem $(1 - s \cos.\eta)^{-\frac{3}{2}}$ in seriem evolvi oportet. Modo ergo communi adhibito reperiemus:

$$(1 + s\cos.\eta)^{-\frac{3}{2}} = 1 + \frac{3}{2} s \cos.\eta + \frac{3.5}{2.4} ss \cos.\eta^2 + \frac{3.5.7}{2.4.6} s^3 \cos.\eta^3 + \frac{3.5.7.9}{2.4.6.8} + s^4 \cos.\eta^4 \ \&c.$$

quæ series, dummodo $s \cos.\eta$ fuerit unitate minus, uti quidem semper usu venit, certe convergit. Interim tamen nisi $s \cos.\eta$ sit valde parvum nimis lente convergit, quam ut aliquot terminis colligendis ejus summa satis exacte obtineri queat. Verum hic quoque commode accidit, ut dum hæc series per sequentes integrationes tractatur, multo promtiorem convergentiam acquirat, quod nisi eveniret, non perspicerem quomodo quæstioni propositæ satisfieri posset; necessitas tunc cogeret investigationi mutationum horariarum unice adhærescere, illasque pro quantumvis magnis temporis intervallis in unam summam colligere.

§. LXVI. Verum cum potestates ipsius $\cos.\eta$ in differentiale $d\eta$ ductæ integrari nequeant, nisi prius in cosinus angulorum multiplorum 2η, 3η, &c. convertantur, quoniam differentiale $d\eta$ ad nostrum commune differentiale $d\omega$ reducere licet, eadem conditio postulat, ut hujus $\cos.\eta$ potestates in cosinus angulorum multiplorum resolvantur id quod ope noti lemmatis: $\cos.\alpha . \cos.\beta = \frac{1}{2}\cos.(\alpha + \beta) + \frac{1}{2}\cos.(\alpha - \beta)$ facile præstatur, reperietur enim

$$\cos.\eta = \cos.\eta,$$
$$\cos.\eta^2 = \frac{1}{2}\cos.2\eta + \frac{1}{2},$$
$$\cos.\eta^3 = \frac{1}{4}\cos.3\eta + \frac{3}{4}\cos.\eta,$$
$$\cos.\eta^4 = \frac{1}{8}\cos.4\eta + \frac{4}{8}\cos.2\eta + \frac{3}{8},$$
$$\cos.\eta^5 = \frac{1}{16}\cos.5\eta + \frac{5}{16}\cos.3\eta + \frac{10}{16}\cos.\eta,$$
$$\cos.\eta^6 = \frac{1}{32}\cos.6\eta + \frac{6}{32}\cos.4\eta + \frac{15}{32}\cos.2\eta + \frac{10}{32},$$
$$\cos.\eta^7 = \frac{1}{64}\cos.7\eta + \frac{7}{64}\cos.5\eta + \frac{21}{64}\cos.3\eta + \frac{35}{64}\cos.\eta,$$
$$\cos.\eta^8 = \frac{1}{128}\cos.8\eta + \frac{8}{128}\cos.6\eta + \frac{28}{128}\cos.4\eta + \frac{56}{128}\cos.2\eta + \frac{35}{128}, \ \&c.$$

§. LXVII. Quodsi hi valores substituantur, obtinebitur $(1 - s\cos.\eta)^{-\frac{3}{2}} =$

$$1 + \frac{3}{2}.s\cos.\eta + \frac{3.5}{2.4}.\frac{1}{2}ss\cos.2\eta + \frac{3.5.7}{2.4.6}.\frac{1}{4}.s^3\cos.3\eta + \frac{3.5.7.9}{2.4.6.8}.\frac{1}{8}s^4\cos.4\eta,$$
$$+ \frac{3.5}{2.4}.\frac{1}{2}ss + \frac{3.5.7}{2.4.6}.\frac{3}{4}s^3\cos.\eta + \frac{3.5.7.9}{2.4.6.8}.\frac{4}{8}s^4\cos.2\eta,$$
$$+ \frac{3.5.7.9}{2.4.6.8}.\frac{3}{8}s^4, \ \&c.$$

Unde si ponamus $(1 - s\cos\eta)^{-\frac{3}{2}} = P + Q s\cos\eta + R ss\cos 2\eta + S s^3 \cos 3\eta + T s^4 \cos 4\eta +$, &c. hi coëfficientes assumti ita definientur ut sit:

$$P = 1 + \tfrac{3\cdot 5}{2\cdot 4}\cdot\tfrac{1}{2}ss + \tfrac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}\cdot\tfrac{3}{8}s^4 + \tfrac{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}\cdot\tfrac{10}{32}s^6 + \text{\&c.}$$

$$Q = \tfrac{3}{2}\left(1 + \tfrac{5\cdot 7}{4\cdot 6}\cdot\tfrac{3}{4}ss + \tfrac{5\cdot 7\cdot 9\cdot 11}{4\cdot 6\cdot 8\cdot 10}\cdot\tfrac{10}{16}s^4 + \tfrac{5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14}\cdot\tfrac{35}{64}s^6 + \text{\&c.}\right)$$

$$R = \tfrac{3\cdot 5}{2\cdot 4}\left(\tfrac{1}{2} + \tfrac{7\cdot 9}{6\cdot 8}\cdot\tfrac{4}{8}ss + \tfrac{7\cdot 9\cdot 11\cdot 13}{6\cdot 8\cdot 10\cdot 12}\cdot\tfrac{15}{32}s^4 + \tfrac{7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17}{6\cdot 8\cdot 10\cdot 12\cdot 14\cdot 16}\cdot\tfrac{56}{128}s^6 + \text{\&c.}\right)$$

$$S = \tfrac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\left(\tfrac{1}{4} + \tfrac{9\cdot 11}{8\cdot 10}\cdot\tfrac{5}{16}ss + \tfrac{9\cdot 11\cdot 13\cdot 15}{8\cdot 10\cdot 12\cdot 14}\cdot\tfrac{21}{64}s^4 + \text{\&c.}\right)$$

$$T = \tfrac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}\left(\tfrac{1}{8} + \tfrac{11\cdot 13}{10\cdot 12}\cdot\tfrac{6}{32}ss + \text{\&c.}\right)$$

$$V = \tfrac{3\cdot 5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8\cdot 10}\left(\tfrac{1}{16} + \text{\&c.}\right)$$

§. LXVIII. Hinc igitur habebimus:

$$\frac{1}{z^3} = \frac{1}{r^3}\left(P + Qs\cos\eta + Rss\cos 2\eta + Ss^3\cos 3\eta + Ts^4\cos 4\eta + \text{\&c.}\right)$$

atque series pro P, Q, R, S, &c., inventæ satis convergunt, ut hi valores per consuetas methodos approximando erui queant. Sequenti modo autem in alias formas transfundi possunt, quæ aptiores videntur

$$P = 1 + \tfrac{3\cdot 5}{4\cdot 4}ss + \tfrac{3\cdot 5\cdot 7\cdot 9}{4\cdot 4\cdot 8\cdot 8}s^4 + \tfrac{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}{4\cdot 4\cdot 8\cdot 8\cdot 12\cdot 12}s^6 + \text{\&c.}$$

$$\tfrac{1}{2}Q = \tfrac{3}{4}\left(1 + \tfrac{5\cdot 7}{4\cdot 8}ss + \tfrac{5\cdot 7\cdot 9\cdot 11}{4\cdot 8\cdot 8\cdot 12}\cdot s^4 + \tfrac{5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15}{4\cdot 8\cdot 8\cdot 12\cdot 12\cdot 16}s^6 + \text{\&c.}\right)$$

$$\tfrac{1}{2}R = \tfrac{3\cdot 5}{4\cdot 4}\left(\tfrac{1}{2} + \tfrac{7\cdot 9}{8\cdot 8}\cdot\tfrac{2}{3}ss + \tfrac{7\cdot 9\cdot 11\cdot 13}{8\cdot 8\cdot 12\cdot 12}\cdot\tfrac{3}{4}s^4 + \tfrac{7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17}{8\cdot 8\cdot 12\cdot 12\cdot 16\cdot 16}\cdot\tfrac{4}{5}s^6 + \text{\&c.}\right)$$

$$\tfrac{1}{2}S = \tfrac{3\cdot 5\cdot 7}{4\cdot 4\cdot 8}\left(\tfrac{1}{3} + \tfrac{9\cdot 11}{8\cdot 12}\cdot\tfrac{2}{4}ss + \tfrac{9\cdot 11\cdot 13\cdot 15}{8\cdot 12\cdot 12\cdot 16}\cdot\tfrac{3}{5}s^4 + \tfrac{9\cdot 11\cdot 13\cdot 15\cdot 17\cdot 19}{8\cdot 12\cdot 12\cdot 16\cdot 16\cdot 20}\cdot\tfrac{4}{6}s^6 + \text{\&c.}\right)$$

$$\tfrac{1}{2}T = \tfrac{3\cdot 5\cdot 7\cdot 9}{4\cdot 4\cdot 8\cdot 8}\left(\tfrac{1\cdot 2}{3\cdot 4} + \tfrac{11\cdot 13}{12\cdot 12}\cdot\tfrac{2\cdot 3}{4\cdot 5}ss + \tfrac{11\cdot 13\cdot 15\cdot 17}{12\cdot 12\cdot 16\cdot 16}\cdot\tfrac{3\cdot 4}{5\cdot 6}s^4 + \tfrac{11\cdot 13\cdot 15\cdot 17\cdot 19\cdot 21}{12\cdot 12\cdot 16\cdot 16\cdot 20\cdot 20}\cdot\tfrac{4\cdot 5}{6\cdot 7}s^6 + \text{\&c.}\right)$$

$$\tfrac{1}{2}V = \tfrac{3\cdot 5\cdot 7\cdot 9\cdot 11}{4\cdot 4\cdot 8\cdot 8\cdot 12}\left(\tfrac{1\cdot 2}{4\cdot 5} + \tfrac{13\cdot 15}{12\cdot 16}\cdot\tfrac{2\cdot 3}{5\cdot 6}\cdot ss + \tfrac{13\cdot 15\cdot 17\cdot 19}{12\cdot 16\cdot 16\cdot 20}\cdot\tfrac{3\cdot 4}{6\cdot 7}s^4 + \text{\&c.}\right)$$

$$\tfrac{1}{2}X = \tfrac{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}{4\cdot 4\cdot 8\cdot 8\cdot 12\cdot 12}\left(\tfrac{1\cdot 2\cdot 3}{4\cdot 5\cdot 6} + \tfrac{15\cdot 17}{16\cdot 16}\cdot\tfrac{2\cdot 3\cdot 4}{5\cdot 6\cdot 7}ss + \tfrac{15\cdot 17\cdot 19\cdot 21}{16\cdot 16\cdot 20\cdot 20}\cdot\tfrac{3\cdot 4\cdot 5}{6\cdot 7\cdot 8}s^4 + \text{\&c.}\right)$$

§. LXIX. Nunc perspicuum est singulas has series in infinitum continuatas fieri geometricas, denominatore existente ss; quare eæ si per $1 - ss$ multiplicentur, multo magis convergentes reddentur. Hoc modo consequemur:

$$P(1-ss) = 1 - \tfrac{1\cdot 1}{4\cdot 4}ss - \tfrac{1\cdot 1\cdot 3\cdot 5}{4\cdot 4\cdot 8\cdot 8}s^4 - \tfrac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9}{4\cdot 4\cdot 8\cdot 8\cdot 12\cdot 12}s^6 - \text{\&c.}$$

$$\tfrac{1}{2}Q(1-ss) = \tfrac{3}{4}\left(1 + \tfrac{1\cdot 3}{4\cdot 8}ss + \tfrac{1\cdot 3\cdot 5\cdot 7}{4\cdot 8\cdot 8\cdot 12}s^4 + \tfrac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{4\cdot 8\cdot 8\cdot 12\cdot 12\cdot 16}s^6 + \text{\&c.}\right)$$

$$\tfrac{1}{2}R(1-ss) = \tfrac{3\cdot 5}{4\cdot 4}\left(\tfrac{1}{2} + \tfrac{3\cdot 5}{8\cdot 8}\cdot\tfrac{2}{3}ss + \tfrac{3\cdot 5\cdot 7\cdot 9}{8\cdot 8\cdot 12\cdot 12}\cdot\tfrac{3}{4}s^4 + \tfrac{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}{8\cdot 8\cdot 12\cdot 12\cdot 16\cdot 16}\cdot\tfrac{4}{5}s^6 + \text{\&c.}\right)$$

$$\tfrac{1}{2}S(1-ss) = \tfrac{3\cdot 5\cdot 7}{4\cdot 4\cdot 8}\left(\tfrac{1}{3} + \tfrac{5\cdot 7}{8\cdot 12}\cdot\tfrac{2}{4}ss + \tfrac{5\cdot 7\cdot 9\cdot 11}{8\cdot 12\cdot 12\cdot 16}\cdot\tfrac{3}{5}s^4 + \tfrac{5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15}{8\cdot 12\cdot 12\cdot 16\cdot 16\cdot 20}\cdot\tfrac{4}{6}s^6 + \text{\&c.}\right)$$

$$\tfrac{1}{2}T(1-ss) = \tfrac{3\cdot 5\cdot 7\cdot 9}{4\cdot 4\cdot 8\cdot 8}\left(\tfrac{1\cdot 2}{3\cdot 4} + \tfrac{7\cdot 9}{12\cdot 12}\cdot\tfrac{2\cdot 3}{4\cdot 5}ss + \tfrac{7\cdot 9\cdot 11\cdot 13}{12\cdot 12\cdot 16\cdot 16}\cdot\tfrac{3\cdot 4}{5\cdot 6}s^4 + \text{\&c.}\right)$$

$$\tfrac{1}{2}V(1-ss) = \tfrac{3\cdot 5\cdot 7\cdot 9\cdot 11}{4\cdot 4\cdot 8\cdot 8\cdot 12}\left(\tfrac{1\cdot 2}{4\cdot 5} + \tfrac{9\cdot 11}{12\cdot 16}\cdot\tfrac{2\cdot 3}{5\cdot 6}ss + \tfrac{9\cdot 11\cdot 13\cdot 15}{12\cdot 16\cdot 16\cdot 20}\cdot\tfrac{3\cdot 4}{6\cdot 7}s^4 + \text{\&c.}\right)$$

$$\tfrac{1}{2}X(1-ss) = \tfrac{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}{4\cdot 4\cdot 8\cdot 8\cdot 12\cdot 12}\left(\tfrac{1\cdot 2\cdot 3}{4\cdot 5\cdot 6} + \tfrac{11\cdot 13}{16\cdot 16}\cdot\tfrac{2\cdot 3\cdot 4}{5\cdot 6\cdot 7}ss + \text{\&c.}\right)$$

$$\tfrac{1}{2}Y(1-ss) = \tfrac{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15}{4\cdot 4\cdot 8\cdot 8\cdot 12\cdot 12\cdot 16}\left(\tfrac{1\cdot 2\cdot 3}{5\cdot 6\cdot 7} + \tfrac{13\cdot 15}{16\cdot 20}\cdot\tfrac{2\cdot 3\cdot 4}{6\cdot 7\cdot 8}ss + \text{\&c.}\right)$$

$$\tfrac{1}{2}Z(1-ss) = \tfrac{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17}{4\cdot 4\cdot 8\cdot 8\cdot 12\cdot 12\cdot 16\cdot 16}\left(\tfrac{1\cdot 2\cdot 3\cdot 4}{5\cdot 6\cdot 7\cdot 8} + \tfrac{15\cdot 17}{20\cdot 20}\cdot\tfrac{2\cdot 3\cdot 4\cdot 5}{6\cdot 7\cdot 8\cdot 9}ss + \text{\&c.}\right)$$

&c.

§. LXX. At vero non opus est ut singuli isti valores evolvantur, sufficit enim duos priores collegisse, ex quibus reliqui per sequentes formulas facile formari poterunt.

$$R = \frac{4Q - 3\cdot 2P}{ss};\quad S = \frac{8R - 5Q}{3ss};$$

$$T = \frac{12S - 7R}{5ss};\quad V = \frac{16T - 9S}{7ss},\ \text{\&c.}$$

Quin etiam ex prima derivari potest secunda, si integrationem in subsidium vocare velimus, est enim

$$Q = 2P - \tfrac{1}{ss}\int Ps\,ds.$$

Sin autem quæratur valor primæ seriei, dico eum per integrationem inveniri posse, sumendo s constans, & introducendo variabilem z foreque:

$$P(1-ss) = \frac{\int\frac{dz}{\sqrt{z}}\left(\frac{1-zz}{ss-zz}\right)^{\frac{1}{4}}}{\int\frac{dz}{\sqrt{z}}\left(\frac{1}{ss-zz}\right)^{\frac{1}{4}}}.$$

Si post integrationem ponatur $z = s$; utrumque autem integrale ita capi sumo, ut evanescat posito $z = o$, quo casu quidem ſit denominator $\int \frac{dz}{\sqrt{z}} \left(\frac{1}{ss-zz}\right)^{\frac{1}{4}} = \frac{\pi}{\sqrt{2}}$, denotante hic π peripheriam, cujus diameter $= 1$.

§. LXXI. Si ergo eſſet $s = 1$, quo caſu hæ ſeries minime convergerent, foret ob numeratorem noſtræ expreſſionis integralis $= \int \frac{dz}{\sqrt{z}} = 2\sqrt{z} = 2\sqrt{s} = 2$, primæ ſeries $P(1-ss) = \frac{2\sqrt{2}}{\pi} = 0,9003162$, quæ ſumma per conſuetas approximandi methodos non difficulter erueretur, unde patet ejus ſummam multo facilius obtineri ſi valor ipſius s uti ſemper evenit ſit unitate minor. In genere autem erit

$$P(1-ss) = \frac{\sqrt{2}}{\pi} \int \frac{dz}{\sqrt{z}} \left(\frac{1-zz}{ss-zz}\right)^{\frac{1}{4}}$$

poſito poſt integrationem $z = s$: quæ expreſſio eo magis eſt notatu digna, quod ejus veritas inveſtiganti non tam cito occurrit. Interim tamen expediet quovis caſu, quo valor ipſius s datur, aliquot terminis harum ſerierum actu addendis earum ſummas prope veras colligere, quod negotium eo promtius ſuccedet quo minor fuerit numerus s. Hoc igitur modo inventis quantitatibus P, Q, R, S, &c. erit

$$\frac{1}{z^3} = \frac{1}{r^3}(P + Qs\,cos.\eta + Rss\,cos.2\eta + Ss^3\,cos.3\eta + Ts^4\,cos.4\eta + \&c)$$

brevitatis gratia autem poſuimus:

$$r = \sqrt{(xx+yy)}, \;\&\; s = \frac{2xy}{xx+yy}.$$

§. LXXII. Pro faciliori autem quantitatum P, Q, R, S, &c. computo conveniet ſerierum illarum coëfficientes infractiones decimales transformari, unde obtinebitur, ſubſcriptis logarithmis

$P(1-ss)$	$\frac{1}{2}Q(1-ss)$	$\frac{1}{2}R(1-ss)$	$\frac{1}{2}S(1-ss)$
$+ 1,000000$	$+ 0,750000$	$+ 0,468750$	$+ 0,273437$
0, 0000000	9, 8750613	9, 6709413	9, 4368580
$- 0,062500.ss$	$+ 0,070313.ss$	$+ 0,146490.ss$	$+ 0,149536.ss$
8, 7958800	8, 8470326	9, 1657913	9, 1747461
$- 0,014648.s^4$	$+ 0,025635.s^4$	$+ 0,072098.s^4$	$+ 0,092525.s^4$
8, 1657913	8, 4088294	8, 8579220	8, 9662614
$- 0,006409.s^6$	$+ 0,013218.s^6$	$+ 0,044983.s^6$	$+ 0,061647.s^6$
7, 8067694	8, 1211634	8, 6530468	8, 7969035
$- 0,003580.s^8$	$+ 0,008055.s^8$	$+ 0,018527.s^8$	$+ 0,045167.s^8$
7, 5538655	7, 9060481	8, 4552557	8, 6548281
$- 0,002282.s^{10}$	$+ 0,005420.s^{10}$	$+ 0,020325.s^{10}$	$+ 0,034088.s^{10}$
7, 3583457	7, 7340094	8, 3080408	8, 5325953
$- 0,001581.s^{12}$	$+ 0,003896.s^{12}$	$+ 0,015217.s^{12}$	$+ 0,026631.s^{12}$
7, 1988962	7, 5905873	8, 1823473	8, 4253853
$- 0,001159.s^{14}$	$+ 0,002935.s^{14}$	$+ 0,011821.s^{14}$	
7, 0642480	7, 4675831	8, 0726486	
$- 0,000887.s^{16}$	$+ 0,002290.s^{16}$		
6, 9477098	7, 3598899		
$- 0,000700.s^{12}$			
6, 8449800			

SECTIO V.

Evolutio formularum differentialium in ſeries ſecundum ſinus coſinusve angulorum ſimpliciter progredientes.

§. LXXIII. Quoniam defectus Analyſeos neceſſitatem nobis imponit omnia integralia, quibus opus eſt, per ſeries ſecundum ſinus coſinusve angulorum ſimpliciter progredientes exprimendi, ſimilem formam ſingulis formulis differentialibus induci opportet. Cum igitur primum pro planeta perturbante ſit $y = \frac{c}{1 - e\, cos.\, u}$, erit terminos qui quadratum excentricitatis e altioresque poteſtates involvunt omittendo:

$$y = c\,(1 + e\, cos.\, u)\,;\ \&\ \frac{1}{y^2} = \frac{1}{cc}\,(1 - 2\,e\, cos.\, u).$$

Deinde cum ſit $d\theta = du = \frac{a\, d\omega \sqrt{ac}}{yy}$, habebimus

$$d\theta = du = \frac{a\sqrt{a}}{c\sqrt{c}}\, d\omega\,(1 - 2\,e\, cos.\, u).$$

Quia enim hujus motus ratio tantum in perturbationes ingreditur non opus eſt has formulas accuratius evolvere.

§. LXXIV. Simili modo cum pro planeta perturbato ſit $x = \frac{p}{1 - q\, cos.\, v}$, erit:

$$\frac{1}{xx} = \frac{1}{pp}\,(1 + \tfrac{1}{2}\,qq - 2\,q\, cos.\, v + \tfrac{1}{2}\,qq\, cos.\, 2\,v),$$

in qua nihil eſt neglectum. Hac igitur erit utendum, ubi non proprie quæſtio circa perturbationes verſatur, ex hac enim formula, etiamſi nulla contingeret perturbatio,

batio, motus planetæ regularis deduci deberet. Id quod evenit in definiendo elemento motus veri $d\varphi$, quod partim ſequitur leges Kepplerianas, partim vero minimis inæqualitatibus perturbatur. Illo reſpectu verus ejus valor capi debebit, qui erit

$$d\varphi = \frac{a\sqrt{a}}{p\sqrt{p}}\, d\omega\,(1 + \tfrac{1}{2}\,qq - 2\,q\, cos.\, v + \tfrac{1}{2}\,qq\, cos.\, 2\,v).$$

Quatenus vero idem valor $d\varphi$ ad ſolas perturbationes inveſtigandas adhibetur, ſufficiet tam pro ſemiparametro p quam pro excentricitate q valores medios conſtantes b & k uſurpare, atque adeo quadratam ipſius k rejicere, ita ut pro hoc uſu habeamus:

$$d\varphi = \frac{a\sqrt{a}}{b\sqrt{b}}\, d\omega\,(1 - 2\,k\, cos.\, v).$$

§. LXXV. Quia formulæ $\frac{a\sqrt{a}}{b\sqrt{b}}$ & $\frac{a\sqrt{a}}{c\sqrt{c}}$ calculum maxime ingrediuntur ponamus brevitatis gratia:

$$\frac{a\sqrt{a}}{b\sqrt{b}} = i,\ \&\ \frac{a\sqrt{a}}{c\sqrt{c}} = m;$$

ſicque erit pro formulis perturbationes implicantibus:

$$d\theta = du = m\, d\omega\,(1 - 2\,e\, cos.\, u),\ \&$$
$$d\varphi = i\, d\omega\,(1 - 2\,k\, cos.\, v).$$

Unde cum angulus $\varphi - \theta$ nonniſi in hoc negotio occurrat, quoniam poſuimus $\varphi - \theta = \eta$, & differentiale anguli η hinc ad differentiale $d\omega$ revocatur. Fiet enim

$$d\eta = (i - m)\, d\omega - 2\,ik\, d\omega\, cos.\, v + 2\,me\, d\omega\, cos.\, u,$$

ubi notandum eſt eſſe:

$i : 1$ ut motus medius planetæ perturbati ad motum medium Solis vel Terræ;

$m : 1$ ut motus medius planetæ perturbantis ad motum medium Solis vel Terræ.

Quare si perturbationes in motu terræ investigentur erit $i = 1$. Verum pro toto motu planetæ perturbati erit:

$$d\varphi = \frac{b\sqrt{b}}{p\sqrt{p}}\, i\, d\omega \left(1 + \tfrac{1}{2} qq - 2q \cos. \nu + \tfrac{1}{2} qq \cos. 2\nu\right).$$

§. LXXVI. Nunc agrediamur valorem $\frac{1}{r^3}$, qui quoniam tantum in perturbationibus inest, minores ejus particulas negligere licebit. Quia igitur pro eo posuimus $\sqrt{(xx + yy)} = r$, erit $\frac{1}{r^3} = (xx + yy)^{-\frac{3}{2}}$; $(bb(1 + 2k \cos. \nu) + cc(1 + 2e \cos. u))^{-\frac{3}{2}}$, ideoque

$$\frac{1}{r^3} = \frac{1}{(bb + cc)^{\frac{3}{2}}} - \frac{3bbk \cos \nu - 3cce \cos. u}{(bb + cc)^{\frac{5}{2}}}.$$

Ponamus brevitatis gratia $\sqrt{(bb + cc)} = f$, ut sit:

$$\frac{1}{r^3} = \frac{1}{f^3} - \frac{3bbk \cos. \nu}{f^5} - \frac{3cce \cos. u}{f^5}.$$

Deinde cum itidem posuerimus $s = \frac{2xy}{xx + yy}$; fiet per easdem positiones:

$$s = \frac{2bc(1 + k \cos. \nu)(1 + e \cos. u)}{ff + 2bbk \cos. \nu + 2cce \cos. u},$$

quæ expressio pari modo evoluta evadet:

$$s = \frac{2bc}{ff} + \frac{2bc(cc - bb)k \cos. \nu}{f^4} - \frac{2bc(cc - bb)e \cos. u}{f^4}.$$

§. LXXVII. Quo etiam hanc formulam commodiorem reddamus, statuamus:

$\frac{2bc}{ff} = \frac{2bc}{bb + cc} = \mu$ & $\frac{cc - bb}{ff} = \frac{cc - bb}{bb + cc} = \nu$, ut $\mu\mu + \nu\nu = 1$; erit $\frac{bb}{ff} = \frac{1 - \nu}{2}$ & $\frac{cc}{ff} = \frac{1 + \nu}{2}$. His autem valoribus introductis, habebimus

$\frac{1}{r^3} = \frac{1}{f^3}\left(1 - \frac{3}{2}(1 - \nu) k \cos. \nu - \frac{3}{2}(1 + \nu) e \cos. u\right)$, &

$s = \mu + \mu\nu k \cos. \nu - \mu\nu e \cos. u = \mu(1 + \nu k \cos. \nu - \nu e \cos. u)$,

unde pro litteris P, Q, R, &c. fit

$$s^2 = \mu^2 (1 + 2\nu k \cos. \nu - 2\nu e \cos. u);$$
$$s^3 = \mu^3 (1 + 3\nu k \cos. \nu - 3\nu e \cos. u);$$
$$s^4 = \mu^4 (1 + 4\nu k \cos. \nu - 4\nu e \cos. u);$$
$$s^5 = \mu^5 (1 + 5\nu k \cos. \nu - 5\nu e \cos. u);$$
&c.

tam vero porro:

$1 - ss = \nu\nu - 2\mu^2 \nu k \cos. \nu + 2\mu^2 \nu e \cos. u$, hincque

$$\frac{1}{1 - ss} = \frac{1}{\nu\nu} + \frac{2\mu^2 k \cos. u}{\nu^3} - \frac{2\mu^2 e \cos. u}{\nu^3}.$$

§. LXXVIII. Si his valoribus adhibitis formulas §. LXXII evolutas ad calculum revocemus, & quantitates P, Q, R, S, &c. investigemus, eæ sequenti modo expressæ reperientur:

$$\frac{P}{r^3} = \frac{1}{f^3}(g + hk \cos. \nu + le \cos. u);$$
$$\frac{Qs}{r^3} = \frac{1}{f^3}(g' + h'k \cos. \nu + l'e \cos. u);$$
$$\frac{Rss}{r^3} = \frac{1}{f^3}(g'' + h''k \cos. \nu + l''e \cos. u);$$
$$\frac{Ss^3}{r^3} = \frac{1}{f^3}(g''' + h'''k \cos. \nu + l'''e \cos. u);$$
&c.

quovis enim casu valores idonei pro g, h, l, g', h', l', &c. per merum calculum numericum reperientur, quos ergo numeros tanquam cognitos spectare licebit. Hinc itaque elicimus $\frac{1}{r^3} = \frac{1}{f^3} \times$

$$\left\{\begin{array}{l} + g \quad + g' \cos. \eta \quad + g'' \cos. 2\eta \quad + g''' \cos. 3\eta \\ + hk \cos. \nu + \frac{1}{2} h'k \cos.(\eta - \nu) + \frac{1}{2} h''k \cos.(2\eta - \nu) + \frac{1}{2} h'''k \cos.(3\eta - \nu), \\ \qquad + \frac{1}{2} h'k \cos.(\eta + \nu) + \frac{1}{2} h''k \cos.(2\eta + \nu) + \frac{1}{2} h'''k \cos.(3\eta + \nu), \\ + le \cos. u + \frac{1}{2} l'e \cos.(\eta - u) + \frac{1}{2} l''e \cos.(2\eta - u) + \frac{1}{2} l'''e \cos.(3\eta - u), \\ \qquad + \frac{1}{2} l'e \cos.(\eta + u) + \frac{1}{2} l''e \cos.(2\eta + u) + \frac{1}{2} l'''e \cos.(3\eta + u), \\ \qquad\qquad \&c. \end{array}\right\}$$

§. LXXIX. Nunc paulatim ad valores litterarum M & N definiendos procedere possumus. Cum enim sit $M = \frac{y\,\mathrm{sin.}\,\eta}{z^3} - \frac{\mathrm{sin.}\,\eta}{yy}$, primo habebimus

$$\frac{y\,\mathrm{sin.}\,\eta}{z^3} = \frac{1}{cc}\left(\mathrm{sin.}\,\eta - e\,\mathrm{sin.}\,(\eta - u) - e\,\mathrm{sin.}\,(\eta + u)\right).$$

Deinde ob $y\,\mathrm{sin.}\,\eta = c\left(\mathrm{sin.}\,\eta + \tfrac{1}{2}e\,\mathrm{sin.}\,(\eta - u) + \tfrac{1}{2}e\,\mathrm{sin.}\,(\eta + u)\right)$, erit $\frac{y\,\mathrm{sin.}\,\eta}{z^3} = \frac{c}{f^3}\times$

$$\left\{\begin{array}{llll}
+g\,\mathrm{sin.}\,\eta & +\tfrac{1}{2}g'\,\mathrm{sin.}\,2\eta & +\tfrac{1}{2}g''\,\mathrm{sin.}\,3\eta & +\tfrac{1}{2}g'''\,\mathrm{sin.}\,4\eta \\
+\tfrac{1}{2}ge\,\mathrm{sin.}\,(\eta-u) & +\tfrac{1}{4}g'e\,\mathrm{sin.}\,(2\eta-u) & -\tfrac{1}{2}g''\,\mathrm{sin.}\,\eta & -\tfrac{1}{2}g'''\,\mathrm{sin.}\,2\eta \\
+\tfrac{1}{2}ge\,\mathrm{sin.}\,(\eta+u) & +\tfrac{1}{4}g'e\,\mathrm{sin.}\,(2\eta+u) & -\tfrac{1}{4}g''e\,\mathrm{sin.}\,(\eta+u) & -\tfrac{1}{4}g'''e\,\mathrm{sin.}\,(2\eta+u) \\
+\tfrac{1}{2}hk\,\mathrm{sin.}\,(\eta-v) & +\tfrac{1}{4}h'k\,\mathrm{sin.}\,(2\eta-v) & +\tfrac{1}{4}g''e\,\mathrm{sin.}\,(3\eta-u) & +\tfrac{1}{4}g'''e\,\mathrm{sin.}\,(4\eta-u) \\
+\tfrac{1}{2}hk\,\mathrm{sin.}\,(\eta+v) & +\tfrac{1}{4}h'k\,\mathrm{sin.}\,(2\eta+v) & -\tfrac{1}{4}g''e\,\mathrm{sin.}\,(\eta-u) & -\tfrac{1}{4}g'''e\,\mathrm{sin.}\,(2\eta-u) \\
+\tfrac{1}{2}le\,\mathrm{sin.}\,(\eta-u) & +\tfrac{1}{4}l'e\,\mathrm{sin.}\,(2\eta-u) & +\tfrac{1}{4}g''e\,\mathrm{sin.}\,(3\eta+u) & +\tfrac{1}{4}g'''e\,\mathrm{sin.}\,(4\eta+u) \\
+\tfrac{1}{2}le\,\mathrm{sin.}\,(\eta+u) & +\tfrac{1}{4}l'e\,\mathrm{sin.}\,(2\eta+u) & -\tfrac{1}{4}h''k\,\mathrm{sin.}\,(\eta-v) & -\tfrac{1}{4}h'''k\,\mathrm{sin.}\,(2\eta-v) \\
 & & +\tfrac{1}{4}h''k\,\mathrm{sin.}\,(3\eta-v) & +\tfrac{1}{4}h'''k\,\mathrm{sin.}\,(4\eta-v) \\
 & & -\tfrac{1}{4}h''k\,\mathrm{sin.}\,(\eta+v) & -\tfrac{1}{4}h'''k\,\mathrm{sin.}\,(2\eta+v) \\
 & & +\tfrac{1}{4}h''k\,\mathrm{sin.}\,(3\eta+v) & +\tfrac{1}{4}h'''k\,\mathrm{sin.}\,(4\eta+v) \\
 & & -\tfrac{1}{4}l''e\,\mathrm{sin.}\,(\eta-u) & -\tfrac{1}{4}l'''e\,\mathrm{sin.}\,(2\eta-u) \\
 & & +\tfrac{1}{4}l''e\,\mathrm{sin.}\,(3\eta-u) & +\tfrac{1}{4}l'''e\,\mathrm{sin.}\,(4\eta-u) \\
 & & -\tfrac{1}{4}l''e\,\mathrm{sin.}\,(\eta+u) & -\tfrac{1}{4}l'''e\,\mathrm{sin.}\,(2\eta+u) \\
 & & +\tfrac{1}{4}l''e\,\mathrm{sin.}\,(3\eta+u) & +\tfrac{1}{4}l'''e\,\mathrm{sin.}\,(4\eta+u)
\end{array}\right\}$$

&c.

§. LXXX. Omittendis jam terminis, qui plusquam triplum anguli η involvunt colligemus valorem ipsius $M = +\frac{c}{f^3}\times$

$$\left\{\begin{array}{llll}
+(g-\tfrac{1}{2}g'')\,\mathrm{sin.}\,\eta & +\tfrac{1}{2}e(g-\tfrac{1}{2}g'')\,\mathrm{sin.}\,(\eta-u) & +e(g'-g''')\,\mathrm{sin.}\,(2\eta-u) & +\tfrac{1}{4}e(g''-g'''')\,\mathrm{sin.}\,(3\eta-u) \\
 & +\tfrac{1}{2}e(g-\tfrac{1}{2}g'')\,\mathrm{sin.}\,(\eta+u) & +e(g'-g''')\,\mathrm{sin.}\,(2\eta+u) & +\tfrac{1}{4}e(g''-g'''')\,\mathrm{sin.}\,(3\eta+u) \\
+\tfrac{1}{2}(g'-g''')\,\mathrm{sin.}\,2\eta & +\tfrac{1}{2}e(l-\tfrac{1}{2}l'')\,\mathrm{sin.}\,(\eta-u) & +e(l'-l''')\,\mathrm{sin.}\,(2\eta-u) & +\tfrac{1}{4}e(l''-l'''')\,\mathrm{sin.}\,(3\eta-u) \\
 & +\tfrac{1}{2}e(l-\tfrac{1}{2}l'')\,\mathrm{sin.}\,(\eta+u) & +e(l'-l''')\,\mathrm{sin.}\,(2\eta+u) & +\tfrac{1}{4}e(l''-l'''')\,\mathrm{sin.}\,(3\eta+u) \\
+\tfrac{1}{2}(g''-g'''')\,\mathrm{sin.}\,3\eta & +\tfrac{1}{2}k(h-\tfrac{1}{2}h'')\,\mathrm{sin.}\,(\eta-v) & +k(h'-h''')\,\mathrm{sin.}\,(2\eta-v) & +\tfrac{1}{4}k(h''-h'''')\,\mathrm{sin.}\,(3\eta-v) \\
\text{\&c.} & +\tfrac{1}{2}k(h-\tfrac{1}{2}h'')\,\mathrm{sin.}\,(\eta+v) & +k(h'-h''')\,\mathrm{sin.}\,(2\eta-v) & +\tfrac{1}{4}k(h''-h'''')\,\mathrm{sin.}\,(3\eta+v)
\end{array}\right\}$$

$$-\frac{1}{cc}\left(\mathrm{sin.}\,\eta - e\,\mathrm{sin.}\,(\eta-u) - e\,\mathrm{sin.}\,(\eta+u)\right).$$

&c.

In qua expressione lex est manifesta, cujus ope plures termini, si quis laborem suscipere velit, formari pos-

sunt. Si quidem numerus $\mu = \frac{2bc}{bb+cc}$ est valde parvus, quod evenit, si quantitates b & c multum à ratione æqualitatis recedunt, vix ultra litteras g, h, l una virgula notatas progredi est opus; etsi autem eæ quantitates propius ad æqualitatem accedunt, tamen calculum vix ultra binas virgulas continuari est opus, quia per integrationem hæ series admodum redduntur convergentes. Ob eandem causam multo magis terminos per ee, kk & ek affectos rejicere licuerat, prætterquam quod excentricitatem utramque e & k valde parvam assuminius.

§. LXXXI. Deinde cum sit $N = \frac{x - y\,\mathrm{cos.}\,\eta}{z^3} + \frac{\mathrm{cos.}\,\eta}{yy}$ habebimus primo

$$\frac{\mathrm{cos.}\,\eta}{yy} = \frac{1}{cc}\left(\mathrm{cos.}\,\eta - e\,\mathrm{cos.}\,(\eta - u) - e\,\mathrm{cos.}\,(\eta + u)\right),$$

porro vero, ob $x = b\,(1 + k\,\mathrm{cos.}\,v)$, &

$$y\,\mathrm{cos.}\,\eta = c\left(\mathrm{cos.}\,\eta + \tfrac{1}{2}e\,\mathrm{cos.}\,(\eta - u) + \tfrac{1}{2}e\,\mathrm{cos.}\,(\eta + u)\right),$$

erit valor ipsius $N = \frac{1}{cc}\left(\mathrm{cos.}\,\eta - e\,\mathrm{cos.}\,(\eta - u) - e\,\mathrm{cos.}\,(\eta + u)\right)$.

$$+\frac{b}{f^3}\left\{\begin{array}{lll}
+g & +g'\,\mathrm{cos.}\,\eta & +g''\,\mathrm{cos.}\,2\eta \\
+k(g+h)\,\mathrm{cos.}\,v & +\tfrac{1}{2}k(g'+h')\,\mathrm{cos.}\,(\eta-v) & +\tfrac{1}{2}k(g''+h'')\,\mathrm{cos.}\,(2\eta-v) \\
+le\,\mathrm{cos.}\,u & +\tfrac{1}{2}k(g'+h')\,\mathrm{cos.}\,(\eta+v) & +\tfrac{1}{2}k(g''+h'')\,\mathrm{cos.}\,(2\eta+v) \\
 & +\tfrac{1}{2}l'e\,\mathrm{cos.}\,(\eta-u) & +\tfrac{1}{2}l''e\,\mathrm{cos.}\,(\eta-u) \\
 & +\tfrac{1}{2}l'e\,\mathrm{cos.}\,(\eta+u) & +\tfrac{1}{2}l''e\,\mathrm{cos.}\,(\eta+u)
\end{array}\right\}$$

$$-\frac{c}{f^3}\left\{\begin{array}{lll}
 & +(g+\tfrac{1}{2}g'')\,\mathrm{cos.}\,\eta & +\tfrac{1}{2}(g'+g''')\,\mathrm{cos.}\,2\eta \\
+\tfrac{1}{2}g' & +\tfrac{1}{2}k(h+\tfrac{1}{2}h'')\,\mathrm{cos.}\,(\eta-v) & +\tfrac{1}{4}k(h'+h''')\,\mathrm{cos.}\,(2\eta-v) \\
 & +\tfrac{1}{2}k(h+\tfrac{1}{2}h'')\,\mathrm{cos.}\,(\eta+v) & +\tfrac{1}{4}k(h'+h''')\,\mathrm{cos.}\,(2\eta+v) \\
+\tfrac{1}{2}h'k\,\mathrm{cos.}\,v & +\tfrac{1}{2}e(g+\tfrac{1}{2}g'')\,\mathrm{cos.}\,(\eta-u) & +\tfrac{1}{4}e(g'+g''')\,\mathrm{cos.}\,(2\eta-u) \\
 & +\tfrac{1}{2}e(g+\tfrac{1}{2}g'')\,\mathrm{cos.}\,(\eta+u) & +\tfrac{1}{4}e(g'+g''')\,\mathrm{cos.}\,(2\eta+u) \\
+\tfrac{1}{2}e(g'+l')\,\mathrm{cos.}\,u & +\tfrac{1}{2}e(l+\tfrac{1}{2}l'')\,\mathrm{cos.}\,(\eta-u) & +\tfrac{1}{4}e(l'+l''')\,\mathrm{cos.}\,(2\eta-u) \\
 & +\tfrac{1}{2}e(l+\tfrac{1}{2}l'')\,\mathrm{cos.}\,(\eta+u) & +\tfrac{1}{4}e(l'+l''')\,\mathrm{cos.}\,(2\eta+u)
\end{array}\right\}$$

§. LXXXII. Excentricitas planetæ perturbantis e has formulas imprimis tantopere reddit prolixas, quæ si evanefceret, hi valores commodius & fuccinctius ita exprimerentur

$$M = -\tfrac{1}{ii}\,\text{fin.}\,\eta + \tfrac{c}{f^3}\begin{Bmatrix} +(g-\tfrac{1}{2}g'')\,\text{fin.}\,\eta + \tfrac{1}{2}k(h-\tfrac{1}{2}h'').\begin{Bmatrix}\text{fin.}(\eta-\nu)\\ \text{fin.}(\eta+\nu)\end{Bmatrix} \\ +\tfrac{1}{2}(g'-g''')\,\text{fin.}\,2\eta + \tfrac{1}{4}k(h'-h''').\begin{Bmatrix}\text{fin.}(2\eta-\nu)\\ \text{fin.}(2\eta+\nu)\end{Bmatrix}\end{Bmatrix}$$

Deinde vero eft:

$$N = \tfrac{1}{cc}\,\text{cof.}\,\eta + \tfrac{b}{f^3}\begin{Bmatrix} +g & +k(g+h)\,\text{cof.}\,\nu \\ +g'\,\text{cof.}\,\eta & +\tfrac{1}{2}k(g'+h').\begin{Bmatrix}\text{cof.}(\eta-\nu)\\ \text{cof.}(\eta+\nu)\end{Bmatrix} \\ +g''\,\text{cof.}\,2\eta & +\tfrac{1}{2}k(g''+h'').\begin{Bmatrix}\text{cof.}(2\eta-\nu)\\ \text{cof.}(2\eta+\nu)\end{Bmatrix}\end{Bmatrix}$$

$$-\tfrac{c}{f^3}\begin{Bmatrix} +\tfrac{1}{2}g' & +\tfrac{1}{2}h'k\,\text{cof.}\,\nu \\ +(g+\tfrac{1}{2}g'')\,\text{cof.}\,\eta & +\tfrac{1}{2}k(h+\tfrac{1}{2}h'').\begin{Bmatrix}\text{cof.}(\eta-\nu)\\ \text{cof.}(\eta+\nu)\end{Bmatrix} \\ +\tfrac{1}{2}(g+g''')\,\text{cof.}\,2\eta & +\tfrac{1}{4}k(h'+h''').\begin{Bmatrix}\text{cof.}\,2\eta-\nu)\\ \text{cof.}\,2\eta+\nu)\end{Bmatrix}\end{Bmatrix}$$

Attendenti autem mox patebit hanc excentricitatem e fine errore fenfibili negligi poffe, unde his poftremis formulis utemur.

§. LXXXIII. His jam valoribus pro M & N inventis ipfa differentialia quibus perturbationes continentur ad formam defideratam reducere poterimus. Quod igitur primum ad variabilitatem femiparametri p attinet, quoniam invenimus $dp = -2nMaxd\omega\sqrt{ap}$ in hac expreffione ob n minima loco p ejus valorem medium b & loco x valorem $b(1+k\,\text{cof.}\,\nu)$ fcribere licebit, unde fit:

$$dp = -2nabd\omega.\ M(1+k\,\text{cof.}\,\nu)\sqrt{ab},\ \text{feu}$$
$$\tfrac{dp}{b} = -2na\sqrt{ab}.\ M(1+k\,\text{cof.}\,\nu)\,d\omega.$$

Cum autem pofuerimus $a\sqrt{a} = ib\sqrt{b}$, erit $\frac{dp}{b} = -2nibbMd\omega(1+k\,\text{cof.}\,\nu)$; hincque pro M valorem inventum fubftituendo

$$\tfrac{dp}{b} = +\tfrac{2nibb}{cc}\,d\omega\left(\text{fin.}\,\eta + \tfrac{1}{2}k\,\text{fin.}(\eta-\nu) + \tfrac{1}{2}k\,\text{fin.}(\eta+\nu)\right);$$

$$-\tfrac{2nibbc}{f^3}\,d\omega.\begin{Bmatrix} +(g-\tfrac{3}{2}g'')\,\text{fin.}\,\eta + \tfrac{1}{2}k(g-\tfrac{1}{2}g''+h-\tfrac{1}{2}h'').\begin{Bmatrix}\text{fin.}(\eta-\nu)\\ \text{fin.}(\eta+\nu)\end{Bmatrix} \\ +\tfrac{1}{2}(g'-g''')\,\text{fin.}\,2\eta + \tfrac{1}{4}k(g'-g'''+h'-h''').\begin{Bmatrix}\text{fin.}(2\eta-\nu)\\ \text{fin.}(2\eta+\nu)\end{Bmatrix}\end{Bmatrix}$$

ubi ex denominationibus factis eft $\frac{bb}{cc} = \frac{m}{i}\sqrt[3]{\frac{m}{i}} = \frac{1-\nu}{1+\nu}$, & $\frac{2bc}{ff} = \mu$, hincque ob $\frac{b}{f} = \sqrt{\frac{1-\nu}{2}}$, erit $\frac{2bbc}{f^3} = \mu\sqrt{\frac{1-\nu}{2}}$. Quare ex cognita ratione motuum mediorum habebitur

$$\mu = \frac{2\sqrt[3]{iimm}}{\sqrt[3]{i^4}+\sqrt[3]{m^4}};\quad \nu = \frac{\sqrt[3]{i^4}-\sqrt[3]{m^4}}{\sqrt[3]{i^4}+\sqrt[3]{m^4}}.$$

§. LXXXIV. Pro variabilitate autem excentricitatis q, quia ea quoque eft minima, in ejus expreffione ponamus itidem $p=b$ & $q=k$, & quoniam terminos qui quadratum kk continerent, negligimus erit ob $a\sqrt{a} = ib\sqrt{b}$; $dq = nibbd\omega(M(2\,\text{cof.}\,\nu - \tfrac{1}{2}k + \tfrac{1}{2}k\,\text{cof.}\,2\nu) + N\,\text{fin.}\,\nu)$; unde pro M & N fubftitutis valoribus obtinebitur:

$$dq = \frac{-nibb}{cc}\,d\omega\Big(\tfrac{1}{2}\text{fin.}(\eta-\nu) + \tfrac{1}{2}\text{fin.}(\eta+\nu) - \tfrac{1}{2}k\,\text{fin.}\,\eta + \tfrac{1}{4}k\,\text{fin.}(\eta-2\nu) + \tfrac{1}{4}k\,\text{fin.}(\eta+2\nu)\Big)$$

$$+\frac{nib^3}{f^3}\,d\omega\begin{Bmatrix} +g\,\text{fin.}\,\nu \begin{matrix}+\tfrac{1}{2}g'\,\text{fin.}(\eta+\nu) + \tfrac{1}{2}g''\,\text{fin.}(2\eta+\nu)\\ -\tfrac{1}{2}g'\,\text{fin.}(\eta-\nu) - \tfrac{1}{2}g''\,\text{fin.}(2\eta-\nu)\end{matrix} \\ +\tfrac{1}{2}k(g+h)\,\text{fin.}\,2\nu \begin{matrix}+\tfrac{1}{4}k(g'+h')\,\text{fin.}(\eta+2\nu) + \tfrac{1}{4}k(g''+h'')\,\text{fin.}(2\eta+2\nu)\\ -\tfrac{1}{4}k(g'+h')\,\text{fin.}(\eta-2\nu) - \tfrac{1}{4}k(g''+h'')\,\text{fin.}(2\eta-2\nu)\end{matrix}\end{Bmatrix}$$

$$+\frac{nibc}{f^3}d\omega\left\{\begin{array}{ll}-\frac{1}{2}g'\sin.\nu & -\frac{1}{4}h'k\sin.2\nu+\frac{1}{4}k(4h-2h''-2g+g'')\sin.\eta\\ +(\frac{1}{2}g-\frac{1}{4}g'')\sin.(\eta-\nu) & +\frac{1}{8}k(6h-h''+2g-g'')\sin.(\eta-2\nu)\\ +(\frac{1}{2}g-\frac{1}{4}g'')\sin.(\eta+\nu) & +\frac{1}{8}k(2h-3h''+2g-g'')\sin.(\eta+2\nu)\\ +\frac{1}{4}(3g'-g''')\sin.(2\eta-\nu) & +\frac{1}{4}k(2h'-2h'''-g'+g''')\sin.2\eta\\ +\frac{1}{4}(g'-3g''')\sin.(2\eta+\nu) & +\frac{1}{8}k(3h'-h'''+g'-g''')\sin.(2\eta-2\nu)\\ & +\frac{1}{8}k(h-3h'''+g-g''')\sin.(2\eta+2\nu)\end{array}\right\}$$

§. LXXXV. Pro motu aphelii autem habebimus negligendis ſimili modo minimis terminis, & pro $a\sqrt{a}$ ſcribendo $ib\sqrt{b}$;

$$d\varphi-d\nu=\frac{nibbd\omega}{f^3k}(M(2\sin.\nu+\tfrac{1}{2}k\sin.2\nu)-N\cos.\nu));$$

quæ expreſſio, ſi loco M & N valores eruti ſubſtituantur, abibit in formam ſequentem

$$\frac{d\varphi-d\nu}{d\omega}=-\frac{nibb}{cck}\Big(\tfrac{1}{2}\cos.(\eta-\nu)-\tfrac{1}{2}\cos.(\eta+\nu)+$$
$$\tfrac{1}{4}k\cos.(\eta-2\nu)-\tfrac{1}{4}k\cos.(\eta+2\nu)\Big)$$

$$-\frac{nib^3}{f^3k}\left\{\begin{array}{ll}+g\cos.\nu & +\frac{1}{2}k(g'+h')\cos.\eta\\ +\frac{1}{2}g'\cos.(\eta-\nu)+\frac{1}{2}k(g+h) & +\frac{1}{4}k(g'+h')\cos.(\eta-2\nu)\\ +\frac{1}{2}g'\cos.(\eta+\nu) & +\frac{1}{4}k(g'+h')\cos.(\eta+2\nu)\\ +\frac{1}{2}g''\cos.(2\eta-\nu)+\frac{1}{2}k(g+h)\cos.2\nu & +\frac{1}{2}k(g''+h'')\cos.2\eta\\ +\frac{1}{2}g''\cos.(2\eta+\nu) & +\frac{1}{4}k(g''+h'')\cos.(2\eta-2\nu)\\ & +\frac{1}{4}k(g''+h'')\cos.(2\eta+2\nu)\end{array}\right\}$$

$$+\frac{nibbc}{f^3k}\left\{\begin{array}{ll}+\frac{1}{2}g'\cos.\nu & +\frac{1}{4}k(2h+h'')\cos.\eta\\ +\frac{1}{4}(6g-g'')\cos.(\eta-\nu)+\frac{1}{4}h'k & +\frac{1}{8}k(6h-h''-2g-g'')\cos.(\eta-2\nu)\\ +\frac{1}{4}(2g-3g'')\cos.(\eta+\nu) & -\frac{1}{8}k(2h-3h''+2g-g'')\cos.(\eta+2\nu)\\ +\frac{1}{4}(3g'-g''')\cos.(2\eta-\nu)+\frac{1}{4}h'k\cos.2\nu & +\frac{1}{4}k(h'+h''')\cos.2\eta\\ -\frac{1}{4}(g'-3g''')\cos.(2\eta+\nu) & +\frac{1}{8}k(3h'-h'''+g'-g''')\cos.2\eta-2\nu)\\ & -\frac{1}{8}k(h(h'-3h'''+g'-g''')\cos.(2+2\nu)\end{array}\right\}$$

§. LXXXVI. Reſtat ut ſimili modo variationes, quibus cum longitudo lineæ nodorum π, tum inclinatio G ſunt obnoxiæ, exprimamus: Ac neglecta quidem excentricitate planetæ perturbantis e, ut ſit $y=c$ & $x=b(1+k\cos.\nu)$, erit $d\pi=-nibbcd\omega$ $(1+k\cos.\nu)(\frac{1}{z^3}-\frac{1}{c^3})\sin.(\varphi-\pi)\sin.(\theta-\pi)$; $d.\,l\,tang.$

$d.\,l\,tang.\,\rho=-nibbcd\omega(1+k\cos.\nu)(\frac{1}{z^3}-\frac{1}{c^3})$ $\cos.(\varphi-\pi)\sin.(\theta-\pi)$; ubi valor ipſius $\frac{1}{z^3}$ debet ſubſtitui qui eſt poſito $e=0$

$$\frac{1}{z^3}=\frac{1}{f^3}\left\{\begin{array}{l}+g+hk\cos.\nu\\ +g'\cos.\eta+\frac{1}{2}h'k\cos.(\eta-\nu)+\frac{1}{2}h''k\cos.(2\eta-\nu)\\ +g''\cos.2\eta+\frac{1}{2}h'k\cos.(\eta+\nu)+\frac{1}{2}h''k\cos.(2\eta+\nu)\end{array}\right\}$$

Tum vero ob $\varphi-\theta=\eta$ eſt

$$\sin.(\varphi-\pi)\sin.(\theta-\pi)=\tfrac{1}{2}\cos.\eta-\tfrac{1}{2}\cos.(\varphi+\theta-2\pi)$$
$$\cos.(\varphi-\pi)\sin.(\theta-\pi)=-\tfrac{1}{2}\sin.\eta+\tfrac{1}{2}\sin.(\varphi+\theta-2\pi).$$

§. LXXXVII. Introducamus ad has formulas aliquanto ſimpliciores reddendas, argumentum latitudinis $\varphi-\pi$, ponamuſque $\varphi-\pi=\sigma$, eritque

$$d\pi=-nibbcd\omega(\tfrac{1}{z^3}-\tfrac{1}{c^3})\left\{\begin{array}{l}\frac{1}{2}\cos.\eta-\frac{1}{2}\cos.(\eta-2\sigma)\\ +\frac{1}{4}k\cos.(\eta-\nu)-\frac{1}{4}k\cos.(\eta-2\sigma-\nu)\\ +\frac{1}{4}k\cos.(\eta+\nu)-\frac{1}{4}k\cos.(\eta-2\sigma+\nu)\end{array}\right\}$$

& ſubſtituto pro $\frac{1}{z^3}$ valore:

$$d\pi=+\frac{nibb}{cc}d\omega\left\{\begin{array}{l}\frac{1}{2}\cos.\eta-\frac{1}{2}\cos.(\eta-2\sigma)\\ +\frac{1}{4}k\cos(\eta-\nu)-\frac{1}{4}k\cos.(\eta-2\sigma-\nu)\\ +\frac{1}{4}k\cos.(\eta+\nu)-\frac{1}{4}k\cos.(\eta-2\sigma+\nu)\end{array}\right\}$$

$$-\frac{nibbc}{f^3}d\omega\left\{\begin{array}{ll}+\frac{1}{4}g' & +\frac{1}{8}k(2h+h''+2g+g'')\cos.(\eta-\nu)\\ +\frac{1}{4}(2g+g''\cos.\eta & +\frac{1}{8}k(2h+h''+2g+g'')\cos.(\eta+\nu)\\ +\frac{1}{4}(g'+g''')\cos.2\eta & +\frac{1}{4}k(h'+g')\cos.\nu\\ -\frac{1}{2}g\cos.(\eta-2\sigma) & +\frac{1}{8}k(h'+h'''+g'+g''')\cos.(2\eta-\nu)\\ -\frac{1}{4}g'\cos.2\sigma & +\frac{1}{8}k(h'+h'''+g'+g''')\cos.(2\eta+\nu)\\ -\frac{1}{4}g'\cos.(2\eta-2\sigma) & -\frac{1}{8}k(2h+2g)\cos(\eta-2\sigma-\nu)\\ -\frac{1}{4}g''\cos.(\eta+2\sigma) & -\frac{1}{8}k(2h+2g)\cos.(\eta-2\sigma+\nu)\\ & -\frac{1}{8}k(h''+g'')\cos.(\eta+2\sigma-\nu)\\ & -\frac{1}{8}k(h''+g'')\cos.(\eta+2\sigma+\nu)\\ & -\frac{1}{8}k(h'+g')\cos.(2\sigma-\nu)\\ & -\frac{1}{8}k(h'+g')\cos.(2\sigma+\nu)\\ & -\frac{1}{8}k(h'+g')\cos.(2\eta-2\sigma-\nu)\\ & -\frac{1}{8}k(h'+g')\cos.(2\eta-2\sigma+\nu)\end{array}\right\}$$

§. LXXXVIII. Simili vero modo æquatio differentialis pro inclinationis variatione erit:

$$d.l\,tang.\rho = n\,i\,b\,b\,c\,d\omega\left(\tfrac{1}{r^3}-\tfrac{1}{c^3}\right)\left\{\begin{matrix}\tfrac{1}{2}\text{ſin.}\,\eta+\tfrac{1}{2}\text{ſin.}\,(\eta-2\sigma)\\ +\tfrac{1}{4}k\,\text{ſin.}(\eta-\nu)+\tfrac{1}{4}k\,\text{ſin.}(\eta-2\sigma-\nu)\\ +\tfrac{1}{4}k\,\text{ſin.}(\eta+\nu)+\tfrac{1}{4}k\,\text{ſin.}(\eta-2\sigma+\nu)\end{matrix}\right\}$$

hincque ob $d.\,l\,tang.\,\rho = \frac{d.\,tang.\,\rho}{tang.\,\rho}$ obtinebitur

$$\frac{d.\,tang\,\rho}{tang.\,\rho} = -\frac{nibb\,d\omega}{cc}\left\{\begin{matrix}\tfrac{1}{2}\text{ſin.}\,\eta+\tfrac{1}{2}\text{ſin.}\,(\eta-2\sigma)\\ +\tfrac{1}{4}k\,\text{ſin.}(\eta-\nu)+\tfrac{1}{4}k\,\text{ſin.}(\eta-2\sigma-\nu)\\ +\tfrac{1}{4}k\,\text{ſin.}(\eta+\nu)+\tfrac{1}{4}k\,\text{ſin.}(\eta-2\sigma+\nu)\end{matrix}\right\}$$

$$+\frac{nibbc}{f^3}d\omega\left\{\begin{matrix}+\tfrac{1}{4}(2g-g'')\,\text{ſin.}\,\eta & +\tfrac{1}{8}k(2g+2h-g''-h'')\,\text{ſin.}(\eta-\nu)\\ +\tfrac{1}{4}(g'-g''')\,\text{ſin.}\,2\eta & +\tfrac{1}{8}k(2g+2h-g''-h'')\,\text{ſin.}(\eta+\nu)\\ +\tfrac{1}{2}g\,\text{ſin.}\,(\eta-2\sigma) & +\tfrac{1}{8}k(g'+h'-g'''-h''')\,\text{ſin.}(2\eta-\nu)\\ -\tfrac{1}{4}g'\,\text{ſin.}\,2\sigma & +\tfrac{1}{8}k(g'+h'-g'''-h''')\,\text{ſin.}(2\eta+\nu)\\ +\tfrac{1}{4}g'\,\text{ſin.}\,(2\eta-2\sigma) & +\tfrac{1}{4}k(g+h)\,\text{ſin.}\,(\eta-2\sigma-\nu)\\ -\tfrac{1}{4}g''\,\text{ſin.}\,(\eta+2\sigma) & +\tfrac{1}{4}k(g+h)\,\text{ſin.}\,(\eta-2\sigma+\nu)\\ & -\tfrac{1}{8}k(g'+h')\,\text{ſin.}\,(2\sigma-\nu)\\ & -\tfrac{1}{8}k(g'+h')\,\text{ſin.}\,(2\sigma+\nu)\\ & +\tfrac{1}{8}k(g'+h')\,\text{ſin.}\,(2\eta-2\sigma-\nu)\\ & +\tfrac{1}{8}k(g'+h')\,\text{ſin.}\,(2\eta-2\sigma+\nu)\\ & -\tfrac{1}{8}k(g''+h'')\,\text{ſin.}\,(\eta+2\sigma-\nu)\\ & -\tfrac{1}{8}k(g''+h'')\,\text{ſin.}\,(\eta+2\sigma+\nu)\end{matrix}\right\}$$

Si quis vellet has formulas ad plures terminos continuare, lex eſt perſpicua, ſecundum quam hoc opus, quouſque libuerit perfici poſſet, verum pro noſtro inſtituto, ne his quidem terminis exhibitis omnibus indigebimus.

SECTIO VI.

Inveſtigatio inæqualitatum quibus ipſa orbita cujuſque Planetæ ab actione reliquorum Planetarum perturbatur.

§. LXXXIX. Quamvis igitur motus cujuſque Planetæ ab actione reliquorum perturbetur, is nihilo minus ſecundum ellipſin, in cujus alterutro foco Sol verſetur, fieri concipi poteſt, dummodo hæc ellipſis tanquam variabilis tam ratione magnitudinis & ſpeciei quam ratione ſitus lineæ abſidum conſideretur. Atque iſta perturbationum repræſentatio Aſtronomorum inſtituto maxime conveniens videtur, qui dum calculo elliptico jam ſunt aſſueti, huic curvæ inhærere malunt, quam alias curvas magis perplexas in calculum Aſtronomicum admittere. Quod propoſitum cum adeo in luna ſequi ſoleant, etiamſi ejus aberrationes à motu elliptico ſint enormes, id multo magis in motu planetarum principalium retinebitur, quemadmodum etiam Aſtronomi eorum orbitas jam mobiles aſſumſerunt contra indolem motus proprii Keppleriani.

§. XC. Ac primo quidem vidimus parametrum orbitæ cujuſque planetæ ab actione reliquorum continuo immutari. Notari ſcilicet debet ejus valor quidam medius, à quo verus mox in exceſſu mox in defectu diſcrepet; ita valorem medium ſemiparametri orbitæ planetæ, de quo quæritur, hic littera b deſignamus, dum littera p pro quovis tempore ejus valorem verum denotat. Quantum igitur p ob actionem certi alicujus planetæ ab b discrepet, ex æquatione differentiali supra §. LXXXIII evoluta per integrationem definiri poterit, ac si isti effectus, quatenus ab unoquoque planeta in parametrum propositi redundant, seorsim computentur, atque in unam summam colligantur, cognoscetur inversa perturbatio, quæ parametro illi ab actione omnium reliquorum planetarum inducitur, cujus collectionis fundamentum in eo est situm, quod singulæ perturbationes sint quam minimæ.

§. XCI. Totum autem integrationis formulæ §. LXXXIII datæ negotium huc reducitur, ut sequentium formularum simplicium: $d\omega\, sin.\, \eta$; $d\omega\, sin.\, 2\eta$; $d\omega\, sin.\, 3\eta$; $d\omega\, sin.\, (\eta \mp \nu)$; $d\omega\, sin.\, (2\eta \mp \nu)$ &c. integralia definiantur, quæ hac methodo investigo: Primo quia hic excentricitatem planetæ perturbantis negligimus, & motus anomaliæ veræ ν quam minime à motu longitudinis φ differt, si quidem motus aphelii certe est tardissimus, habebimus ex §. LXXV.

$$d\eta = (i - m)\, d\omega - 2\, i\, k\, d\omega\, cos.\, \nu, \;\&$$
$$d\nu = i\, d\omega - 2\, i\, k\, d\omega\, cos.\, \nu.$$

Jam pro prima formula $d\omega\, sin.\, \eta$, differentiale $d\omega$ ita ad $d\eta$ revoco ut sit

$$d\omega = \frac{d\eta}{i-m} + \frac{2\, i\, k\, d\omega}{i-m}\, cos.\, \nu,$$ unde conficitur:

$$d\omega\, sin.\, \eta = \frac{d\eta\, sin.\, \eta}{i-m} + \frac{i\, k\, d\omega}{i-m}\, sin.\, (\eta - \nu) + \frac{i\, k\, d\omega}{i-m}\, sin.\, (\eta + \nu),$$

quo pacto primum membrum jam redditum est integrabile.

§. XCII. Si idem valor pro $d\omega$ etiam in formulis $d\omega\, sin.\, 2\eta$ & $d\omega\, sin.\, 3\eta$ substituatur, erit simili modo

$$d\omega\, sin.\, 2\eta = \frac{d\eta\, sin.\, 2\eta}{i-m} + \frac{i\, k\, d\omega}{i-m}\, sin.\, (2\eta - \nu)$$
$$+ \frac{i\, k\, d\omega}{i-m}\, sin.\, (2\eta + \nu);$$

$$d\omega\, sin.\, 3\eta = \frac{d\eta\, sin.\, 3\eta}{i-m} + \frac{i\, k\, d\omega}{i-m}\, sin.\, (3\eta - \nu)$$
$$+ \frac{i\, k\, d\omega}{i-m}\, sin.\, (3\eta + \nu).$$

Integratis ergo partibus prioribus, habebimus:

$$\int d\omega\, sin.\, \eta = \frac{-\, cos.\, \eta}{i-m} + \frac{i\, k}{i-m}\int d\omega\, sin.\, (\eta - \nu)$$
$$+ \frac{i\, k}{i-m}\int d\omega\, sin.\, (\eta + \nu);$$

$$\int d\omega\, sin.\, 2\eta = \frac{-\, cos.\, 2\eta}{2\,(i-m)} + \frac{i\, k}{i-m}\int d\omega\, sin.\, (2\eta - \nu)$$
$$+ \frac{i\, k}{i-m}\int d\omega\, sin.\, (2\eta + \nu);$$

$$\int d\omega\, sin.\, 3\eta = \frac{-\, cos.\, 3\eta}{3\,(i-m)} + \frac{i\, k}{i-m}\int d\omega\, sin.\, (3\eta - \nu)$$
$$+ \frac{i\, k}{i-m}\int d\omega\, sin.\, (3\eta + \nu).$$

Sicque integrandæ restant reliquæ formulæ, quas nostra expressio pro dp inventa combinet, hæ autem formulæ quia per excentricitatem k sunt multiplicatæ, multo minores sunt prioribus partibus jam integratis, ideoque nisi precisio ultra necessitatem urgeri debeat, satis tuto omitti possent; si quidem jam ob similem caussam excentricitatem e negleximus.

§. XCIII. Interim tamen quo darius perspiciatur, integrationem ex hac parte non impediri, atque pari facilitate perfici posse etiamsi nullos terminos rejecissemus, etiam horum integralia definiam: Pro $\int d\omega\, sin.\, (\eta - \nu)$ igitur quæro primum

$d\eta - d\nu = -m\, d\omega$, ut ſit $d\omega = \frac{-(d\eta - d\nu)}{m}$;

ſicque erit $\int d\omega\, \text{ſin.}\, (\eta - \nu) = \frac{+\,\text{cof.}\, (\eta - \nu)}{m}$.

Deinde pro $\int d\omega\, \text{ſin.}\, (\eta + \nu)$ colligo

$d\eta + d\nu = (2i - m)\, d\omega - 4ik\, d\omega\, \text{cof.}\, \nu$;

unde erit $d\omega = \frac{d\eta + d\nu}{2i - m} + \frac{4ik\, d\omega\, \text{cof.}\, \nu}{2i - m}$, ideoque

$$\int d\omega\, \text{ſin.}\, (\eta + \nu) = \frac{-\text{cof.}\, (\eta + \nu)}{2i - m} + \frac{4ik}{2i - m} \int d\omega\, \text{cof.}\, \nu\, \text{ſin.}\, (\eta + \nu).$$

Sed quia in noſtra formula $\int d\omega\, \text{ſin.}\, (\eta + \nu)$ jam per k eſt multiplicatum, poſterius membrum, quod adhuc integrari deberet, omittimus, qui produceret quantitatem per kk affectam. Hac omiſſione pariter facta pro reliquis formulis, habebimus etiamnunc in differentialibus:

$2d\eta - d\nu = (i - 2m)\, d\omega$, & $2d\eta + d\nu = (3i - 2m)\, d\omega$,

ideoque $d\omega = \frac{2d\eta - d\nu}{i - 2m}$, & $d\omega = \frac{2d\eta + d\nu}{3i - 2m}$.

§. XCIV. His igitur valoribus adhibitis adipiſcemur facile formulas integrales ſequentes:

$$\int d\omega\, \text{ſin.}\, (\eta - \nu) = \frac{+\,\text{cof.}\, (\eta - \nu)}{m};$$

$$\int d\omega\, \text{ſin.}\, (\eta + \nu) = \frac{-\,\text{cof.}\, (\eta + \nu)}{2i - m};$$

$$\int d\omega\, \text{ſin.}\, (2\eta - \nu) = \frac{-\,\text{cof.}\, (2\eta - \nu)}{i - 2m};$$

$$\int d\omega\, \text{ſin.}\, (2\eta - \nu) = \frac{-\,\text{cof.}\, (2\eta + \nu)}{3i - 2m};$$

atque ex his jam priora integralia completa reddentur:

$$\int d\omega\, \text{ſin.}\, \eta = -\frac{\text{cof.}\, \eta}{i - m} + \frac{ik\, \text{cof.}\, (\eta - \nu)}{(i - m)\, m} - \frac{ik\, \text{cof.}\, (\eta + \nu)}{(i - m)(2i - m)},$$

$$\int d\omega\, \text{ſin.}\, 2\eta = -\frac{\text{cof.}\, 2\eta}{2(i - m)} - \frac{ik\, \text{cof.}\, (2\eta - \nu)}{(i - m)(i - 2m)} - \frac{ik\, \text{cof.}\, (2\eta + \nu)}{(i - m)(3i - 2m)},$$

$$\int d\omega\, \text{ſin.}\, 3\eta = -\frac{\text{cof.}\, 3\eta}{3(i - m)} - \frac{ik\, \text{cof.}\, (3\eta - \nu)}{(i - m)(2i - 3m)} - \frac{ik\, \text{cof.}\, (3\eta + \nu)}{(i - m)(4i - 3m)}.$$

&c.

Quæ integralia non ſolum ad valorem integralem ipſius p, ſed etiam ipſius q inveniendum inſerviunt.

§. XCV. Cum nimirum valor medius ipſius p debeat eſſe $= b$, in integratione circa adjectionem conſtantis nullum erit dubium; ſingulis igitur partibus integratis reperietur

$$\frac{p}{b} = 1 - \frac{2nibb}{cc} \left\{ \frac{\text{cof.}\, \eta}{i - m} - \frac{(3i - m)\, k\, \text{cof.}\, (\eta - \nu)}{2(i - m)\, m} + \frac{(3i - m)\, k\, \text{cof.}\, (\eta + \nu)}{2(i - m)(2i - m)} \right\}$$

$$+ \frac{2nibbc}{f^3} \left\{ \begin{aligned} & + \frac{(2g - g'')\, \text{cof.}\, \eta}{2(i - m)} + \frac{ik(2g - g'')\, \text{cof.}\, (\eta - \nu)}{2(i - m)\, m} \\ & + \frac{(g' - g''')\, \text{cof.}\, 2\eta}{4(i - m)} + \frac{k(2g - g'' + 2h - h'')\, \text{cof.}\, (\eta - \nu)}{4m} \\ & + \frac{(g'' - g'''')\, \text{cof.}\, 3\eta}{6(i - m)} - \frac{ik(g' - g''')\, \text{cof.}\, (2\eta - \nu)}{2(i - m)(i - 2m)} \\ & - \frac{k(g' - g''' + h' - h''')\, \text{cof.}\, (2\eta - \nu)}{4(i - 2m)} \\ & - \frac{ik(2g - g'')\, \text{cof.}\, (\eta + \nu)}{2(i - m)(2i - m)} \\ & - \frac{k(2g - g'' + 2h - h'')\, \text{cof.}\, (\eta + \nu)}{4(2i - m)} \\ & - \frac{ik(g' - g''')\, \text{cof.}\, (2\eta + \nu)}{2(i - m)(3i - 2m)} \\ & - \frac{k(g' - g'' + h' - h''')\, \text{cof.}\, (2\eta + \nu)}{4(3i - 2m)} \end{aligned} \right\}$$

Ac si terminos per excentricitatem k affectos, ut pote præ reliquis valde parvos negligamus, erit succinctius:

$$\frac{p}{b} = 1 - \frac{2\,n\,i\,b\,b\,\text{cos.}\,\eta}{(i-m)\,c\,c} + \frac{n\,i\,b\,b\,c}{(i-m)f^3}\Big(\,(2g - g'')\ \text{cos.}\,\eta$$
$$+ \tfrac{1}{2}(g' - g''')\,\text{cos.}\,2\eta + \tfrac{1}{3}(g'' - g'''')\,\text{cos.}\,3\eta + \&c.\Big)$$

ubi notandum esse $\frac{b}{c} = \sqrt[3]{\frac{mm}{ii}}$, & $ff = bb + cc$.

§. XCVI. Ope earundem formularum simplicium integralium etiam vera excentricitas orbitæ q per integrationem differentialis (§. LXXXIV) evoluti assignari poterit; modo adjiciatur $\int d\omega\,\text{sin.}\,v = \int \frac{dv}{i}\,\text{sin.}\,v = \frac{-\text{cos.}\,v}{i}$, si quidem porro ex his expressionibus minimis terminos excentricitatem k involventes negligere pergamus. Hinc igitur posita excentricitate media $= k$, erit excentricitas vera:

$$q = k - \frac{n\,i\,b\,b}{c\,c}\left(\frac{3\,\text{cos.}\,(\eta - v)}{2m} - \frac{\text{cos.}\,(\eta + v)}{2(2i - m)}\right);$$

$$- \frac{n\,i\,b^3}{f^3}\left\{\begin{array}{l} \dfrac{g\,\text{cos.}\,v}{i} + \dfrac{g'\,\text{cos.}\,(\eta - v)}{2m} - \dfrac{g'\,\text{cos.}\,(\eta + v)}{2(2i - m)} \\ \qquad - \dfrac{g''\,\text{cos.}\,(2\eta - v)}{2(i - 2m)} + \dfrac{g''\,\text{cos.}\,(2\eta + v)}{2(3i - 2m)} \end{array}\ \&c. \right\}$$

$$+ \frac{n\,i\,b\,b\,c}{f^3}\left\{\begin{array}{l} + \dfrac{g'\,\text{cos.}\,v}{2i} + \dfrac{(6g - g'')\,\text{cos.}\,(\eta - v)}{4m} - \dfrac{(2g - 3g'')\,\text{cos.}\,(\eta + v)}{4(2i - m)} \\ \qquad - \dfrac{(3g' - g''')\,\text{cos.}\,(2\eta - v)}{4(i - 2m)} - \dfrac{(g' - 2g''')\,\text{cos.}\,(2\eta + v)}{4(3i - 2m)} \end{array}\right\}$$

&c.

Ubi quidem assumimus excentricitatem mediam k tantam esse, ut ejus respectu istæ inæqualitates longe sint minimæ; patet autem has inæqualitates non ab ipsa magnitudine media excentricitatis k pendere, sed easdem prodire sive k sit major sive minor. Quod secus accidit in variationibus lateris recti, quæ sunt proportionales ipsi magnitudini mediæ parametri.

§. XCVII.

§. XCVII. Cognito jam semiparametro p & excentricitate q, semiaxis transversus orbitæ facile definietur, cum sit $= \frac{p}{1 - qq}$. Erit igitur variabilis tam ob variabilitatem ipsius p, quam ipsius q, sed hæc posterior tantum terminos producit per k affectos, unde his neglectis variatio axis transversi potissimum pendebit à variatione parametri, hincque ergo erit

$$\text{Semiaxis transversus} = \frac{b}{1 - kk} - \frac{2\,n\,i\,b\,b}{(i - m)\,c^2}.\ b\,\text{cos.}\,\eta;$$
$$+ \frac{n\,i\,b\,b\,c}{(i - m)f^3}.\ b\left(\,(2g - g'')\ \text{cos.}\,\eta + \tfrac{1}{2}(g' - g''')\right.$$
$$\left.\text{cos.}\,2\eta + \tfrac{1}{3}(g'' - g'''')\ \text{cos.}\,3\eta + \&c.\right)$$

Quare tam parameter & axis transversus quam excentricitas, variationes tantum subeunt periodicas, quæ post certa temporis intervalla ad statum pristinum revertantur, neque perpetua sive incrementa sive decrementa capiunt; sed quantum certis temporibus fuerint aucta, tandundem aliis temporibus diminuentur. Ceterum ex hac applicatione ad axem transversum patet, valorem inventum pro q, etsi terminos tantum primi ordinis continet, tamen æque longe productum esse æstimandum atque valorem ipsius p in quo terminos primi & secundi ordinis evolvimus.

§. XCVIII. Denique definiendus occurrit motus aphelii, in quo præcipuus effectus actionis mutuæ planetarum, quem quidem observationes evidenter manifestarint, cernitur; is autem per integrationem formulæ (§. LXXXV) datæ determinabitur. Aliæ autem hic adsunt formulæ simplices integrandæ, quarum integrationem quoque ad secundum ordinem continuari oportet, uti circa parametrum fecimus, non quo termini secundi ordinis præ primo minus negligi queant, sed quia fecundus ordo continet partes omnino conftantes, unde per integrationem hujufmodi termini $\alpha\,\omega$ nafcuntur, qui quantumvis coëfficiens α fuerit parvus, tamen cum tempore continuo crefcunt. Quia enim angulus ω eft tempori proportionalis, hi termini motum medium aph. lii declarabunt; in quorum idcirco inveftigatione vel minima particula perperam negligitur. At terminis hujus formæ exceptis, reliqui ad fecundum ordinem pertinentes, quia periodicas inæqualitates continent, & præ primo ordine valde funt parvi fine errore omitti poterunt; cum etiam levis error in loco aphelii commiffus nullius fit momenti.

§. XCIX. Simili igitur modo integrationem inftituendo, ante omnia fequentes formulas expendere oportet

$$d\nu = i\,d\omega - 2\,i\,k\,d\omega\,cof.\,\nu;$$

$$d\eta - d\nu = -m\,d\omega - o;$$

$$d\eta + d\nu = (2\,i - m)\,d\omega - 4\,i\,k\,d\omega\,cof.\,\nu;$$

$$2\,d\eta - d\nu = (i - 2\,m)\,d\omega - 2\,i\,k\,d\omega\,cof.\,\nu;$$

$$2\,d\eta + d\nu = (3i - 2\,m)\,d\omega - 6\,i\,k\,d\omega\,cof.\,\nu;$$

$$d\omega = \tfrac{d\nu}{i} + 2\,k\,d\omega\,cof.\,\nu;$$

$$d\omega = \frac{-(d\eta - d\nu)}{m};$$

$$d\omega = \frac{d\eta + d\nu}{2\,i - m} + \frac{4\,i\,k\,d\omega\,cof.\,\nu}{2\,i - m};$$

$$d\omega = \frac{2\,d\eta - d\nu}{i - 2\,m} + \frac{2\,i\,k\,d\omega\,cof.\,\nu}{i - 2\,m};$$

$$d\omega = \frac{2\,d\eta + d\nu}{3\,i - 2\,m} + \frac{6\,i\,k\,d\omega\,cof.\,\nu}{3\,i - 2\,m};$$

tum pro terminis fecundi ordinis:

$$d\omega = \frac{d\nu}{i} = \frac{d\eta}{i - m} = \frac{-(d\eta - 2\,d\nu)}{i + m} = \frac{d\eta + 2\,d\nu}{3\,i - m}$$
$$= \frac{-(2\,d\eta - 2\,d\nu)}{2\,m} = \frac{2\,d\eta + 2\,d\nu}{2\,(2\,i - m)}$$

Hinc omittendis terminis fecundi ordinis, qui non funt formæ $\alpha\,\omega$ fiet $\int d\,\omega\,cof.\,\nu = \frac{fin.\,\nu}{i} + k\int d\,\omega\,(1 + cof.\,2\,\nu) = \frac{fin.\,\nu}{i} + k\,\omega;$

$$\int d\,\omega\,cof.\,(\eta - \nu) = -\frac{fin.\,(\eta - \nu)}{m},$$

$$\int d\,\omega\,cof.\,(2\eta - \nu) = +\frac{fin.\,(2\,\eta - \nu)}{i - 2\,m},$$

$$\int d\,\omega\,cof.\,(\eta + \nu) = \frac{+fin.\,(\eta + \nu)}{2\,i - m},$$

$$\int d\,\omega\,cof.\,(2\,\eta + \nu) = \frac{+fin.\,(2\,\eta + \nu)}{3\,i - 2\,m}.$$

quæ formulæ ad motum aphelii definiendum fufficiunt.

§. C. Ex his igitur differentiale (§. LXXXV) integratum præbebit motum aphelii fequenti modo expreffum:

$$\varphi - \nu = Conft. + \frac{n\,i\,b\,b}{c\,c\,k}\left(\frac{3\,fin.\,(\eta - \nu)}{2\,m} + \frac{fin.\,(\eta + \nu)}{2\,(2\,i - m)}\right)$$

$$- \frac{n\,i\,b^3}{f^3}\left\{\begin{array}{l} + \frac{g\,fin.\,\nu}{i} \quad - \frac{g'\,fin.\,(\eta - \nu)}{2\,m} + \frac{g'\,fin.\,(\eta + \nu)}{2\,(2\,i - m)} \\ + \frac{1}{2}k\,(3\,g + h)\,\omega + \frac{g''\,fin.\,(2\,\eta - \nu)}{2\,(i - 2\,m)} + \frac{g''\,fin.\,(2\,\eta + \nu)}{2\,(3\,i - 2\,m)} \end{array}\right\}$$

$$+ \frac{n\,i\,b\,b\,c}{f^3\,k}\left\{\begin{array}{l} + \frac{g'\,fin.\,\nu}{2\,i} \quad - \frac{(6\,g - g'')\,fin.\,(\eta - \nu)}{4\,m} + \frac{(3\,g' - g''')\,fin.\,(2\,\eta - \nu)}{4\,(i - 2\,m)} \\ + \frac{1}{4}k\,(2\,g' + h')\,\omega - \frac{(2\,g - 3\,g'')\,fin.\,(\eta + \nu)}{4\,(2\,i - m)} - \frac{(g' - 3\,g''')\,fin.\,(2\,\eta + \nu)}{4\,(3\,i - 2\,m)} \end{array}\right\}$$

Hujus expreffionis pars præcipua formæ $\alpha\,\omega$ motum medium aphelii præbet, qui ergo uti perfpicuum eft non à quantitate excentricitatis pendet. Tempore fcilicet quo ſol ſecundum motum medium percurrit angulum $= \omega$ aphelium planetæ proferetur per angulum $\frac{nibbc}{4f^3}(2g' + h')\,\omega - \frac{nib^3}{2f^3}(3g + h)\,\omega$ reliqui vero termini inæqualitates periodicas aphelii complectuntur, quæ eo evadunt majores quo minor fuerit excentricitas orbitæ.

§. CI. Præter hunc autem motum uniformem, quo aphelium profertur, ejus locus ad quodvis tempus corrigi debet per inæqualitates periodicas, quæ ſinibus angulorum $\nu, \eta \mp \nu, 2\eta \mp \nu$, &c. ſunt proportionales: atque in hunc finem longitudo aphelii ita exprimetur:

$$\varphi - \nu = Conſt. + \frac{nibbc}{4f^3}(2g' + h')\,\omega - \frac{nib^3}{2f^3}(3g + h)\,\omega;$$

$$+ \frac{nibb}{2cck}\left(\frac{3\,ſin.(\eta - \nu)}{m} + \frac{ſin.(\eta + \nu)}{2i - m}\right);$$

$$- \frac{nib^3}{4f^3k}\left\{\begin{aligned}&\frac{2g\,ſin.\,\nu}{i} - \frac{g'\,ſin.(\eta - \nu)}{m} + \frac{g'\,ſin.(\eta + \nu)}{2i - m}\\ &+ \frac{g''\,ſin.(2\eta - \nu)}{i - 2m} + \frac{g''\,ſin.(2\eta + \nu)}{3i - 2m}\end{aligned}\right\}$$

$$+ \frac{nibbc}{4f^3k}\left\{\begin{aligned}&\frac{2g'\,ſin.\,\nu}{i} - \frac{(6g - g'')\,ſin.(\eta - \nu)}{m} - \frac{(2g - 3g'')\,ſin.(\eta + \nu)}{2i - m}\\ &+ \frac{(3g' - g''')\,ſin.(2\eta - \nu)}{i - 2m} - \frac{(g' - 3g''')\,ſin.(2\eta + \nu)}{3i - 2m}\end{aligned}\right\}$$

Cujus expreſſionis pars prima exhibet longitudinem mediam aphelii ad quodvis tempus, cui porro ſi applicentur inæqualitates reliqua parte contentæ, impetrabitur locus aphelii verus. Quodſi ponatur $\omega = 360$, ex prima parte innoteſcet motus aphelii annuus reſpectu ſtellarum fixarum.

§. CII. Quia in motu Lunæ inveſtigatio motus ejus apogei tantam diligentiam ac ſagacitatem, totque calculos intricatos exigebat, dubium hic oriri poteſt, an

hoc modo verus motus apheliorum eliciatur? Quodſi enim idem calculus ad Lunam transferretur, formula inventa ſemiſſem tantum veri motus apogei prope modum eſſet oſtenſura. Verum in hac applicatione ad Lunam numerus n ſeu potius termini hunc numerum continentes incomparabiliter prodeunt majores, quam noſtro caſu, atque termini quadratum numeri n involventes demum veram motus apogei quantitatem complent. Hic autem ob valores terminorum numero n affectorum minimos, nullum eſt dubium, quin terminos, qui ejus quadratum complecterentur, ſine ullius erroris ſenſibilis metu prætermittere queamus. Deinde etiam ex formulis generalioribus evidens eſt, excentricitatem planetæ perturbantis e nihil ad motum aphelii conferre.

SECTIO VII.

Inveſtigatio Anomaliæ veræ quatenus ea ad quodvis tempus ab actione Planetarum mutua perturbatur.

§. CIII. In ſuperiori ſectione formulas eruimus, quibus ad quodvis tempus veri valores cum parametri & excentricitatis orbitæ, tum etiam vera longitudo aphelii definiuntur; in has autem formulas præter angulum n potiſſimum ingreditur angulus ν qui planetæ anomaliam veram deſignat. Præcipuum opus igitur adhuc perficiendum in hoc conſiſtit, ut methodum tradamus ad quodvis tempus anomaliam veram inveniendi; quæ cum, ſi nullæ adſint perturbationes ex anomalia media colligi ſoleat; hic quoque anomaliam planetæ mediam in computum introduci conveniet, quæ quoniam uniformiter cum tempore creſcit, ad quodvis tempus tempore creſcit, ad quodvis tempus expedite aſſignatur; ſive quod eodem redit anomalia media reperitur, ſi à planetæ longitudine media, aphelii locus medius ſubtrahatur. Quæſtio ergo hac ſectione enodanda determinationem anomaliæ veræ ν ex data anomalia media poſtulat.

§. CIV. Si nullæ adeſſent vires turbantes, foret $d\varphi = d\nu$, atque anomalia vera ν ex hac æquatione $d\varphi = d\nu = \frac{a\,d\omega\sqrt{ap}}{xx}$, ſeu $d\omega = \frac{p\sqrt{p}}{a\sqrt{a}} \cdot \frac{d\nu}{(1-q\cos.\nu)^2}$ definiri deberet; eſſent enim p & q quantitates conſtantes, & $d\omega$ incremento anomaliæ mediæ proportionale. In noſtro autem caſu neque quantitates p & q ſunt con-

ſtantes, neque $d\varphi = d\nu$, unde manifeſtum eſt relationem inter anomalias mediam & veram quoque ab actione planetarum mutua perturbari. Interim tamen hæc relatio erit petenda ex æquatione $d\varphi = \frac{a\,d\omega\sqrt{ap}}{xx}$, ſeu hæc $d\omega = \frac{p\sqrt{p}}{a\sqrt{a}} \cdot \frac{d\varphi}{(1-q\cos.\nu)^2}$ ſubſtituendo pro $d\varphi$ valorem, qui ipſi ex æquatione differentiali motus aphelii convenit; hæcque æquatio in §. LXXXV habetur evoluta vi cujus cum non ſit $d\varphi - d\nu = 0$; ponamus brevitatis gratia loco hujus æquationis differentialis $d\varphi - d\nu = nV\,d\omega$, eritque

$$d\omega = \frac{p\sqrt{p}}{a\sqrt{a}} \cdot \frac{d\nu}{(1-q\cos.\nu)^2} + \frac{p\sqrt{p}}{a\sqrt{a}} \cdot \frac{nV\,d\omega}{(1-q\cos.\nu)^2}.$$

§. CV. Cum jam p & q non ſint quantitates conſtantes, eorumque valores in ſuperiori ſectione ſint definiti, ponamus quoque brevitatis gratia

$$p = b(1+nP), \ \& \ q = k + nQ;$$

ita ut nP, nQ, & nV ſint effectus perturbationis, eritque ob numerum n minimum $p\sqrt{p} = b^{\frac{3}{2}}(1+\frac{3}{2}nP)$, & quia poſuimus $\frac{a\sqrt{a}}{b\sqrt{b}} = i$ habebimus

$$\frac{p\sqrt{p}}{a\sqrt{a}} = \frac{1}{i}(1+\tfrac{3}{2}nP).$$

Deinde fractio $\frac{1}{(1-q\cos.\nu)^2}$ in ſeriem converſa dat proxime

$$(1-qq)^{-\frac{3}{2}}(1+2q\cos.\nu+\tfrac{3}{2}qq\cos.2\nu+q^3\cos.3\nu\ \&c.)$$

quæ ponendo $k+nQ$ loco q, & negligendo terminos per nn & nkk affectos abit in hanc:

$$(1-kk)^{-\frac{3}{2}}(1+2k\cos.\nu+\tfrac{3}{2}kk\cos.2\nu+k^3\cos.3\nu)$$
$$+2nQ\cos.\nu+3nkQ\cos.2\nu+3nkQ.$$

Hinque erit

$$\frac{p\sqrt{p}}{a\sqrt{a}} \cdot \frac{d\nu}{(1-q\cos.\nu)^2} = \frac{d\nu(1+2k\cos.\nu+\frac{3}{2}kk\cos.2\nu+k^3\cos.3\nu)}{i(1-kk)\sqrt{(1-kk)}}$$

$+\frac{n\,dv}{i}(2Q\cos.\nu+\frac{3}{2}P+3kP\cos.\nu+3kQ+3kQ\cos.2\nu).$

§. CVI. In parte altera autem formulæ integrandæ tam p quam q pro conſtantibus haberi poſſunt, eritque ergo ea pari

$$\frac{n}{i}\,d\omega\,(V+2\,k\,V\cos.\nu),$$

& quia in his particulis minimis eſt $d\nu = i\,d\omega\,(1-2k\cos.\nu)$ obtinebimus æquationem ſequentem:

$$d\omega = \frac{d\nu\,(1+2\,k\cos.\nu+\frac{3}{2}\,kk\cos.2\,\nu+k^3\cos.3\,\nu)}{i\,(1-kk)\,\sqrt{(1-kk)}};$$

$$+\,n\,d\omega\,(2\,Q\cos.\nu+k\,Q+k\,Q\cos.2\,\nu+\tfrac{3}{2}\,P)$$
$$+\,\frac{n\,d\omega}{i}\,(V+2\,k\,V\cos.\nu);$$

cujus pars principalis integrata deducet ad hanc æquationem integralem:

$$\omega = \frac{\nu+2\,k\sin.\nu+\frac{3}{4}\,kk\sin.2\,\nu+\frac{1}{3}\,k^3\sin.3\,\nu}{i\,(1-kk)\,\sqrt{(1-kk)}};$$

$$+\,n\int d\omega\left(\tfrac{3}{2}\,P+2\,Q\cos.\nu+k\,Q+k\,Q\cos.2\,\nu\right.$$
$$\left.+\,\tfrac{1}{i}\,V+\tfrac{2}{i}\,k\,V\cos.\nu\right);$$

cujus poſtremæ partis non amplius erit difficile integrale eruere.

§. CVII. Pro integratione hujus poſtremæ formulæ notandum eſt partem $n\int V\,d\omega$ exprimere motum aphelii, cujus ergo integrale jam ſupra §. CI eſt inventum. Reliquas partes tantiſper indicemus ſigno ſummatorio, ac pro anomalia vera quæſita ſequentem nanciſcemur æquationem:

$$\nu = i(1-kk)^{\frac{3}{2}}\,\omega - 2k\sin.\nu - \tfrac{3}{4}kk\sin.2\nu - \tfrac{1}{3}k^3\sin.3\nu - n\int V\,d\omega;$$
$$-\,i\,n\int d\omega\,(\tfrac{3}{2}\,P+2\,Q\cos.\nu+\tfrac{2}{i}\,k\,V\cos.\nu);$$

in hac enim ultima parte perſpicuum eſt terminos $k\,Q$ & $k\,Q\cos.\nu$ præ P & Q poſſe rejici, at vero $k\,V\cos.\nu$ iiſdem

iiſdem eſſe quaſi homogeneum unde tantum opus eſt valores ſupra pro P, Q, & V inventos ſubſtituere. Hic autem primo obſervo terminum $i\,(1-kk)^{\frac{3}{2}}\omega$ cum partibus formæ $\alpha\,d\omega$, quas poſteriora membra integralia forte continent, deſignare anomaliam mediam, quæ ad quodvis tempus facile colligitur. Si ergo anomaliam mediam ponamus $=\aleph$, habemus hic æquationem inter $\aleph$ & ν, per cujus reſolutionem non difficulter pro quavis anomalia media ejus reſpondens anomalia vera elicietur.

§. CVIII. Statuamus ad abbreviandum:

$$\begin{aligned}\tfrac{3}{2}P+2\,Q\cos.\nu &= A+B\cos.2\,\nu \quad +C\cos.\eta\\ +\tfrac{2}{i}\,k\,V\cos.\nu &\quad +D\cos.2\,\eta \quad +E\cos.(\eta-2\,\nu)\\ &\quad +F\cos.(\eta+2\,\nu)+G\cos.(2\,\eta-2\,\nu)\\ &\quad +H\cos.(2\eta+2\nu)\end{aligned}$$

atque horum coëfficientium valores ex ſuperioribus formulis colliguntur:

$$A = \frac{bbc}{f^3}\cdot g' - \frac{2\,b^3}{f^3}\,g; \quad B = \frac{bbc}{f^3}\cdot g' - \frac{2\,b^3}{f^3}\,g;$$

$$C = \left\{\begin{aligned}&-\frac{ibb}{cc^3}\left(\frac{3}{i-m}+\frac{3}{2\,m}-\frac{1}{2\,(2\,i-m)}+\frac{1}{i}\right)-\frac{i\,b^3}{2\,f^3}\\ &\qquad\left(\frac{2\,g'}{i}+\frac{g'}{m}+\frac{g'}{2\,i-m}\right)\\ &+\frac{ibbc}{4\,f^3}\left(\frac{6\,(2\,g-g'')}{i-m}+\frac{(6\,g-g'')}{m}+\frac{(6\,g-g'')}{i}\right.\\ &\qquad\left.-\frac{(2\,g-3\,g'')}{2\,i-m}-\frac{(2\,g-3\,g'')}{i}\right)\end{aligned}\right\}$$

$$D = \left\{\begin{aligned}&-\frac{i\,b^3}{2\,f^3}\left(\frac{-g''}{i-2\,m}+\frac{2\,g''}{i}+\frac{g''}{3\,i-2\,m}\right)\\ &+\frac{ibbc}{4\,f^3}\left(\frac{3\,(g'-g''')}{i-m}-\frac{(3\,g'-g''')}{i-2\,m}+\frac{(3\,g'-g''')}{i}\right.\\ &\qquad\left.-\frac{(g'-3\,g''')}{3\,i-2\,m}-\frac{(g'-3\,g''')}{i}\right)\end{aligned}\right\}$$

$$E = \frac{-ibb}{2cc}\left(\frac{3}{m} + \frac{3}{i}\right) - \frac{ib^3}{2f^3}\left(\frac{g'}{m} + \frac{g'}{i}\right) + \frac{ibbc}{4f^3}\left(\frac{6g - g''}{m} + \frac{6g - g''}{i}\right)$$

$$F = \frac{-ibb}{2cc}\left(-\frac{1}{2i - m} - \frac{1}{i}\right) - \frac{ib^3}{2f^3}\left(\frac{g'}{2i - m} + \frac{g'}{i}\right) + \frac{ibbc}{4f^3}\left(-\frac{(2g - 3g'')}{2i - m} - \frac{(2g - 3g'')}{i}\right)$$

$$G = \frac{-ib^3}{2f^3}\left(-\frac{g''}{i - 2m} + \frac{g''}{i}\right) + \frac{ibbc}{4f^3}\left(-\frac{(3g' - g''')}{i - 2m} + \frac{(3g' - g''')}{i}\right)$$

$$H = \frac{-ib3}{2f^3}\left(\frac{+g''}{3i - 2m} + \frac{g''}{i}\right) + \frac{ibbc}{4f^3}\left(-\frac{(g' - 3g''')}{3i - 2m} - \frac{(g' - 3g''')}{i}\right)$$

§. CIX. Valoribus autem horum coëfficientum definitis facile erit ſingulorum terminorum integralia exhibere, quia ultra ordinem primum ea deducere non eſt opus: Erit itaque

$$\int d\omega\left(\tfrac{3}{2}P + 2Q\,\mathrm{cof.}\,\nu + \tfrac{2}{i}k\,V\,\mathrm{cof.}\,\nu\right) = A\omega + \frac{B}{2i}\,\mathrm{fin.}\,2\nu + \frac{C}{i-m}\,\mathrm{fin.}\,\eta + \frac{D}{2(i-m)}\,\mathrm{fin.}\,2\eta$$
$$- \frac{E}{i+m}\,\mathrm{fin.}\,(\eta - 2\nu) + \frac{F}{3i-m}\,\mathrm{fin.}\,(\eta + 2\nu) - \frac{G}{2m}\,\mathrm{fin.}\,(2\eta - 2\nu) + \frac{H}{2(2i-m)}\,\mathrm{fin.}\,(2\eta + 2\nu).$$

Deinde ſi ponamus ſimili modo ad abbreviandum

$$\int V\,d\omega = \Delta\omega + \frac{\alpha}{k}\,\mathrm{fin.}\,\nu + \frac{\zeta}{k}\,\mathrm{fin.}\,(\eta - \nu) + \frac{\gamma}{k}\,\mathrm{fin.}\,(\eta + \nu) + \frac{\delta}{k}\,\mathrm{fin.}\,(2\eta - \nu) + \frac{\varepsilon}{k}\,\mathrm{fin.}\,(2\eta + \nu);$$

erunt hi coëfficientes ex §. CI.

$$\Delta = \frac{ibbc}{4f^3}(2g' + h') - \frac{ib^3}{2f^3}(3g + h);$$

$$\alpha = \frac{bbc}{2f^3}\cdot g' - \frac{b^3}{f^3}\cdot g;$$

$$\zeta = \frac{bb}{cc}\cdot\frac{3i}{2m} + \frac{b^3}{2f^3}\cdot\frac{ig'}{m} - \frac{bbc}{4f^3}\cdot\frac{(6g - g'')i}{m};$$

$$\gamma = \frac{bb}{2cc}\cdot\frac{i}{2i - m} - \frac{b^3}{2f^3}\cdot\frac{ig'}{2i - m} - \frac{bbc}{4f^3}\cdot\frac{(2g - 3g'')i}{2i - m};$$

$$\delta = -\frac{b^3}{2f^3}\cdot\frac{ig''}{i - 2m} + \frac{bbc}{4f^3}\cdot\frac{(3g' - g''')i}{i - 2m};$$

$$\varepsilon = -\frac{b^3}{2f^3}\cdot\frac{ig''}{3i - 2m} - \frac{bbc}{4f^3}\cdot\frac{(g - 3g''')i}{3i - 2m}.$$

§. CX. Si jam hos valores determinaverimus, habebimus primo anomaliam mediam:

$$\theta = i(1 - kk)^{\frac{1}{2}}\omega - n\Delta\omega - inA\omega;$$

qua cognita anomalia vera ν ita debet definiri ut ſit:

$$\nu = \theta - 2k\,\mathrm{fin.}\,\nu - \tfrac{3}{4}kk\,\mathrm{fin.}\,2\nu - \tfrac{1}{3}k^3\,\mathrm{fin.}\,3\nu;$$
$$- \frac{n}{k}\left(\alpha\,\mathrm{fin.}\,\nu + \zeta\,\mathrm{fin.}\,(\eta - \nu) + \gamma\,\mathrm{fin.}\,(\eta + \nu) + \delta\,\mathrm{fin.}\,(2\eta - \nu) + \varepsilon\,\mathrm{fin.}\,(2\eta + \nu)\right);$$
$$- n\left\{\begin{array}{l} \tfrac{1}{2}B\,\mathrm{fin.}\,2\nu + \frac{Ci}{i-m}\,\mathrm{fin.}\,\eta + \frac{Di}{2(i-m)}\,\mathrm{fin.}\,2\eta \\ - \frac{Ei}{i+m}\,\mathrm{fin}\,(\eta - 2\nu) + \frac{Fi}{3i-m}\,\mathrm{fin.}\,(\eta + 2\nu) \\ - \frac{Gi}{2m}\,\mathrm{fin.}\,(2\eta - 2\nu) + \frac{Hi}{2(2i-m)}\,\mathrm{fin.}\,(2\eta + 2\nu) \end{array}\right\}$$

Si eſſet $n = o$, nota eſt operatio, qua ex data anomalia media θ elicitur vera ν; cum igitur ſit n fractio valde parva, per eandem operationem, omittendis primum terminis per n affectis quæratur anomalia vera ν mediæ ȣ conveniens, eaque deinceps per terminos fractione n affectos corrigatur. Tum ſi eam accuratius definire velimus, valorem pro ν modo inventum in expreſſione illa pro ν reperta ſubſtituamus, ex eoque denuo ν determinemus.

§. CXI. Facilius autem per conſuetas tabulas anomaliarum totum hoc negotium expediri poteſt. Cum enim perturbationes ſunt minimæ, ſufficiet pro iis anomaliam veram ν proxime ſaltem noſſe, ejuſque ergo loco anomalia media ipſa ȣ uti licebit, namque errores, qui hoc modo committentur, ad ſequentem terminorum, quos negligimus, ordinem pertinerent. Tum valor horum terminorum minimorum ad anomaliam mediam ȣ referatur, ſeu ex data anomalia media ȣ quæratur anomalia media correcta ȣ$'$ ut ſit

$$\text{ȣ}' = \text{ȣ} - \tfrac{n}{k}\Big(\alpha \sin.\text{ȣ} + \beta \sin.(\eta - \text{ȣ}) + \gamma \sin.(\eta + \text{ȣ}) + \delta \sin.(2\eta - \text{ȣ}) + \varepsilon \sin.(2\eta + \text{ȣ})\Big);$$

ubi quidem partem poſteriorem, utpote præ hac valde parvam omitto, atque jam pro data excentricitate k ex tabulis conſuetis quæratur anomalia vera quæ huic anomaliæ mediæ correctæ reſpondeat; hocque modo obtinebitur ipſa illa anomalia vera ν, qua pro evolutione omnium formularum hactenus inventarum indigemus, erit ſcilicet

$$\nu = \text{ȣ}' - 2k \sin.\nu - \tfrac{1}{4} kk \sin.2\nu - \tfrac{1}{3} k^3 \sin.3\nu.$$

§. CXII. In Tabulis autem Aſtronomicis pro data quavis anomalia media non tam ei reſpondens anomalia vera; quam differentia, quæ proſtaphæreſis ſeu æquatio centri vocatur, exhiberi ſolet, neque etiam pro noſtro ſcopo quicquam in hoc inſtituto immutari eſt opus. Ad manus igitur ſit tabula more ſolito adornata, quæ pro excentricitate k cuique anomaliæ mediæ reſpondentem æquationem centri exhibeat. Antequam autem hac tabula utamur, anomalia media planetæ ȣ ad datum tempus collecta, per inæqualitates ſupra expoſitas & tam ab ea ipſa quam ab angulo η, cujus valorem quoque ex motu medio utriuſque planetæ collegiſſe ſufficit, pendentes corrigatur, ut obtineatur anomalia media correcta ȣ$'$. Tum in dicta Tabula quæratur æquatio iſti anomaliæ mediæ ȣ$'$ conveniens, quæ ſit $= \pm Æ$, qua inventa ſtatim habebitur anomalia vera quæſita $\nu = \text{ȣ}' \pm Æ$, qua in determinatione & evolutione omnium formularum ſupra inventarum uti opportebit. Simul vero hæc æquatio $\pm Æ$ ex tabula deſumta verum valorem formulæ $- 2k \sin.\nu - \tfrac{1}{4} kk \sin.2\nu - \tfrac{1}{3} k^3 \sin.3\nu$ exhibebit, id quod pro ſequenti calculo probe notaſſe conducet.

§. CXIII. Ad Anomaliam mediam autem pro dato tempore colligendam motum aphelii medium duntaxat noſſe opportet, qui membro primo formulæ §. CI erutæ continetur, ex quo habemus:

$$\text{Long. aphelii mediam} = Conſt. + \frac{nibbc}{4f^3}(2g' + h')\omega - \frac{nib^3}{2f^3}(3g + h)\omega,$$

ſeu abbreviationem ante introductum adhibendo erit:

$$\text{Long. aphelii media} = Conſt. + n\Delta\omega.$$

Quoniam autem omnes planetæ ad motum medium aphelii aliquid conferunt, ſingulorum effectus exquiri debet, ut inde ad quodvis tempus propoſitum longitudo aphelii media rite obtineatur. Vel cum ex collatione recentiorum obſervationum cum antiquis motus medius aphelii cujuſque planetæ ſatis accurate jam ſit exploratus, eo potius uti conveniet. Quare cum hinc ad tempus propoſitum longitudo media aphelii ſit definita, ea à longitudine media ipſius planetæ ſubtracta præbebit ejus anomaliam mediam ȣ pro eodem tempore propoſito, quæ etiam in nonnullis tabulis immediate exprimi ſolet.

§. CXIV. Deinde ſi perturbationes, quæ ab actione certi cujuſdam planetæ proficiſcuntur, indagare velimus, primum ex collatione ſemi parametri ejus orbitæ c, cum ſemi parametro b planetæ examinandi ope formularum §. LXXII & LXXVIII datarum computentur valores litterarum g, g', g'', &c. ex hiſque porro per rationem mediorum motuum i & m valores litterarum α, β, γ, δ, ε, itemque A, B, C, D, E, F, G, H, qui omnes in meris numeris expreſſi prodibunt. Tum etiam maſſa Planetæ per maſſam Solis diviſa dabit fractionem n. Quibus inventis ad tempus propoſitum colligatur longitudo media planetæ perturbati & perturbantis, quia poſteriori à priori ablata remanebit angulus, quo loco ω uti licebit in indagatione correctionum anomaliæ mediæ (§. CXI). Vel quod expediet, utriuſque planetæ longitudo per tabulas ordinarias definiatur, ac differentia pro angulo ω aſſumatur, quandoquidem hic valor à vero nonniſi in minutiis diſcrepabit.

SECTIO VIII.

Expoſitio Univerſi Calculi quo verus Planetæ locus in orbita ob actionem reliquorum planetarum perturbatus aſſignatur.

§. CXV. PRIMA operatio in hoc conſiſtet, ut pro quolibet planeta, à cujus actione motus planetæ propoſiti perturbatur, ope formularum §. LXXII & LXXVIII expoſitarum primo valores litterarum g, g', g'', &c. (litteris enim reliquis h, h' l, l', &c. ibidem adhibitis carere poſſumus), tum vero ex his porro valores litterarum A, B, C, D, &c. ex §. CVIII, & litterarum quoque Δ, α, β, γ, δ, &c. ex §. CIX per calculum evolvantur: pro quo calculo recordari debemus, fractionem n obtineri, ſi maſſa planetæ perturbantis per maſſam ſolis dividatur: deinde ſi motus diurnus medius ſolis unitate exponatur, exprimet littera i motum diurnum medium planetæ perturbati, & m planetæ perturbantis. Calculus quidem pro valoribus illarum litterarum inſtituendus admodum eſt moleſtus, verumtamen per ſubſidia indicata ſatis exacte abſolvi poterit.

§. CXVI. Statim autem ex valore A cognoſcetur, quantum aphelium ab actione cujuſdam planetæ promoveatur; ſi enim pro angulo ω ponamus 360° terminus $n\Delta\omega$ dabit motum aphelii annuum, ac ſi hunc valorem ab actione cujuslibet planetæ deducamus, omnes conjunctim oſtendent verum motum annuum planetæ reſpectu ſtellarum fixarum, qui vix quicquam ab eo, quem per obſervationes cognovimus, diſcrepare deprehendetur. Cognito autem tam aphelii quam ipſius, planetæ motu medio ad quodvis tempus propoſitum tam hujus planetæ longitudo media quam anomalia media facile aſſignabitur. Statuamus ergo ejus longitudinem mediam $= \zeta$ & anomaliam mediam $= \text{ȣ}$; tum vero excentricitas media ſit $= k$.

§. CXVII. Deinde hæc anomalia media ȣ ex tabulis mediorum motuum deſumta corrigi debet per formulam §. CXI allatam, ut obtineatur anomalia media correcta $\text{ȣ}'$. Vel ſi tabulæ mediorum motuum loco anomaliæ mediæ exhibeant locum aphelii medium, eædem correctiones ſignis verſis ad aphelium applicari debebunt: hoc autem modo reperietur ipſa longitudo aphelii vera, unde hæc correctio magis eſt naturalis priori anomaliæ illata. Quare ex longitudine aphelii media quæratur longitudo ejus vera per hanc formulam

$$\text{Longitudo Aphelii vera} = \text{Longitudini Aphelii mediæ} + \tfrac{n}{k}\Big(\alpha \sin. \text{ȣ} + \beta \sin. (\eta - \text{ȣ}) + \gamma \sin. \eta + \text{ȣ}) + \delta \sin. (2\eta - \text{ȣ}) + \varepsilon \sin. (2\eta + \text{ȣ})\Big);$$

quæ correctio, quia per excentricitatem k eſt diviſa ſatis notabilis eſſe poteſt. Tum iſta Longitudo aphelii vera ſubtrahatur à longitudine planetæ media ζ, ut obtineatur anomalia media ejus correcta $\text{ȣ}'$.

§. CVXIII. Tertio in promtu eſto tabula æquationum centri more ſolito ad excentricitatem k computata, ex qua pro anomalia media $\text{ȣ}'$ excerpatur æquatio centri reſpondens quæ ſit $\pm Æ$, atque hinc reperietur Anomalia vera $= \text{ȣ}' \pm Æ$ quæ ob duplicem cauſam ab anomalia vera, quæ more ſolito ex tabulis æquationum colligitur nullo reſpectu ad perturbationes habito; diſcrepat, primo enim etſi ex eadem tabula deſumta eſt, tamen alii anomaliæ mediæ ac vulgo reſpondet, ideoque

que tantumdem diſcrepat; deinde quia alii anomaliæ mediæ reſpondet, etiam æquatio $\pm Æ$ erit diverſa. Manifeſtum autem eſt hoc poſterius diſcrimen multo fore minus priore; cum hoc adeo eo majus evadat, quo minor fuerit excentricitas k, tum vero æquatio $\pm Æ$ diminuatur. Etſi ergo in calculo perturbationum non adeo accurate noſſe opus eſt anomaliam veram ν, tamen correctio anomaliæ mediæ ſeu loci aphelii neutiquam negligi poteſt.

§. CXIX. Definita hoc modo anomalia vera ν ſtatim locum planetæ in orbita aſſignare poterimus, ita ut non opus habeamus ante variationem parametri & excentricitatis exquirere: quantum enim hæ variationes ad locum planetæ in orbita perturbandum conferunt, id jam ſumus complexi in expreſſione pro loco aphelii vero ſupra §. CI inventa. Nam quia jam valorem ipſius ν exacte expreſſum habemus, erit longitudo vera

$$\varphi = \nu + n \int V\, d\omega + Conſt.$$

Si ergo pro $\int V\, d\omega$ valorem §. CIX poſitum & pro ν valorem §. CX aſſignatum ſubſtituamus, conſequemur

$$\varphi = Conſt. + i(1 - kk)^{\frac{1}{2}}\omega - inA\omega - 2k \sin. \nu - \tfrac{1}{4}kk \sin. 2\nu - \tfrac{1}{3}k^3 \sin. 3\nu - \tfrac{nB}{2} \sin. 2\nu - \tfrac{nCi}{i-m} \sin. \eta - \tfrac{nDi}{2(i-m)} \sin. 2\eta + \tfrac{nEi}{i+m} \sin. (\eta - 2\nu) - \tfrac{nFi}{3i-m} \sin. (\eta + 2\nu) + \tfrac{nGi}{2m} \sin. (2\eta - 2\nu) - \tfrac{nHi}{2(2i-m)} \sin. (2\eta + 2\nu)$$

neque igitur hic amplius inæqualitates illæ majores in forma $n \int V\, d\omega$ contenta aliter ingrediuntur, niſi quatenus illis ipſa anomalia vera ν jam eſt immutata.

§. CXX. Prima portio hujus expreſſionis $i\omega((1 - kk)^{\frac{1}{2}} - nA)$ motum medium hujus planetæ exponit, quem ergo etiam ab actione planetarum aliquantillum perturbari manifeſtum eſt; hinc ſi longitudo planetæ media ponatur $= \zeta$, erit $\zeta = Conſt. + i(1 - kk)^{\frac{3}{2}}\,\omega - i\,n\,A\,\omega$. Deinde vidimus portionem $-2\,k\,ſin.\,\nu - \frac{3}{4}\,kk\,ſin.\,2\,\nu - \frac{1}{3}\,k^3\,ſin.\,3\,\nu$ deſignare æquationem centri $\pm Æ$ quæ in tabulis ordinariis anomaliæ mediæ correctæ ♉' reſpondet, dummodo hæ tabulæ excentricitati k ſint juſto calculo ſuperſtructæ. Cum igitur tam longitudo media ζ quam iſta æquatio $\pm Æ$ conſtet, habebitur longitudo vera planetæ in ſua orbita:

$$\varphi = \zeta \pm Æ - \frac{nB}{2}\,ſin.\,2\,\nu - \frac{n\,C\,i}{i-m}\,ſin.\,\eta - \frac{n\,D\,i}{2(i-m)}\,ſin.\,2\,\eta$$
$$+ \frac{n\,E\,i}{i+m}\,ſin.\,(\eta - 2\,\nu) - \frac{n\,F\,i}{3\,i-m}\,ſin.\,(\eta + 2\,\nu)$$
$$+ \frac{n\,G\,i}{2\,m}\,ſin.\,(2\,\eta - 2\,\nu) - \frac{n\,H\,i}{2(2\,i-m)}\,ſin.\,(2\,\eta + 2\,\nu)$$

ubi portio $\zeta \pm Æ$ exhibet longitudinem modo ordinario inventam, niſi quatenus anomalia media hic eſt correcta, tum vero reliqui termini continent ceteras inæqualitates ab actione planetæ perturbantis profectas, quarum quidem portio quædam jam in ipſa æquatione centri $\pm Æ$ ob anomaliam mediam correctam comprehenditur.

§. CXXI. Actionum ergo planetæ perturbantis ad duplicem effectum perduximus, dum altero longitudo aphelii ſeu anomalia media, altero vero ipſa longitudo perturbatur. Quia vero & priori effectus valde eſt parvus, uterque commode ad unum revocari poterit. Cum enim anomalia vera tantumdem immutetur quantum anomalia media, ſi ν denotet eam ipſam anomaliam veram, quæ anomaliæ mediæ non correctæ ſeu naturali ♉ reſpondet, in expreſſione pro vero planetæ loco inventa, loco ν ſcribi opportet $\nu - \frac{n}{k}\big(\alpha\,ſin.\,\nu + \beta\,ſin.$

$(\eta - \nu) + \gamma\,ſin.\,(\eta + \nu) + \delta\,ſin.\,(2\,\eta - \nu) + \varepsilon\,ſin.\,(2\,\eta + \nu)\big)$ quæ mutatio quidem in terminis minimis nullam variationem ſenſibilem gignit. At ſi jam $\pm Æ$ denotet æquationem centri ipſi anomaliæ mediæ ♉ convenientem, quia eſt $\pm Æ = -2\,k\,ſin.\,\nu - \frac{1}{4}\,kk\,ſin.\,2\,\nu - \frac{1}{3}\,k^3\,ſin.\,3\,\nu$, in primo termino mutatio ſenſibilis orietur, ideoque loco $\pm Æ$ ſcribi debebit

$$\pm Æ + n\big(a\,ſin\,2\,\nu + (\beta + \gamma)\,ſin.\,\eta + (\delta + \varepsilon)\,ſin.\,2\,\eta + \beta\,ſin.\,(\eta - 2\,\nu) + \gamma\,ſin.\,(\eta + 2\,\nu) + \delta\,ſin.\,(2\,\eta - 2\,\nu) + \varepsilon\,ſin.\,(2\,\eta + 2\,\nu)\big);$$

ſicque jam $\pm Æ$ denotabit æquationem centri anomaliæ mediæ naturali ♉ reſpondentem, & anomalia vera ν erit etiam ea quæ more ſolito ſumitur ſcilicet $\nu = ♉ \pm Æ$.

§. CXXII. Hinc igitur faciliorem modum adipiſcimur effectum perturbationis in loco planetæ determinandi. More ſcilicet ſolito ad datum tempus colligatur anomalia media ♉, eique ex tabula ordinaria capiatur reſpondens æquatio centri $\pm Æ$, indeque formetur anomalia vera $\nu = ♉ \pm Æ$. Qua ſtabilita, ſi longitudo planetæ media fuerit $= \zeta$ & η deſignet angulum, qui relinquitur ſi a longitudine planetæ perturbati longitudo planetæ perturbantis ſubtrahatur, habebitur longitudo planetæ perturbati vera.

$$\varphi = \zeta \pm Æ + n\Big\{\Big(\alpha - \frac{B}{2}\Big)ſin.\,2\,\nu + \Big(\beta + \gamma - \frac{C\,i}{i-m}\Big)ſin.\,\eta + \Big(\delta + \varepsilon - \frac{D\,i}{2(i-m)}\Big)ſin.\,2\,\eta$$
$$+ \Big(\beta + \frac{E\,i}{i+m}\Big)ſin.\,(\eta - 2\,\nu) + \Big(\gamma - \frac{F\,i}{3\,i-m}\Big)ſin.\,(\eta + 2\,\nu)$$
$$+ \Big(\delta + \frac{G\,i}{2\,m}\Big)ſin.\,(2\,\eta - 2\,\nu) + \Big(\varepsilon - \frac{H\,i}{2(2\,i-m)}\Big)ſin.\,(2\,\eta + 2\,\nu)\Big\}$$

ubi $\zeta + Æ$ exprimit longitudinem planetæ, quam tabulæ ordinariæ præbent, totaque perturbatio jam in terminis annexis continetur.

§. CXXIII. Hic ſtatim obſervo fieri $\alpha - \frac{B}{2} = 0$, unde ſi ad reliquos terminos contrahendos ponatur:

$$\varphi = \zeta \pm Æ + n B' \text{ſin.}\, \eta + n\, C' \text{ſin.}\, 2\eta + n\, D' \text{ſin.}\, (\eta - 2\nu) + n\, E' \text{ſin.}\, (\eta + 2\nu) + n\, F' \text{ſin.}\, (2\eta - 2\nu) + n\, G' \text{ſin.}\, (2\eta + 2\nu) + \&c.$$

per valores ſupra exhibitos reperiemus

$$B' = \left\{ \begin{array}{l} \frac{bb}{cc} \left(\frac{3 i^3}{m(i-m)^2} - \frac{ii}{(i-m)(2i-m)} \right) + \frac{b^3}{f^3} \cdot \frac{2 i^3 g'}{m(i-m)(2i-m)} \\ - \frac{bbc}{2f^3} \left(\frac{3(2g - g'')ii}{(i-m)^2} + \frac{(6g - g'')ii}{m(i-m)} - \frac{(2g - 3g'')ii}{(i-m)(2i-m)} \right) \end{array} \right\}$$

$$C' = \left\{ \begin{array}{l} - \frac{b^3}{f^3} \cdot \frac{i^3 g''}{(i-m)(i-2m)(3i-2m)} \\ - \frac{bbc}{4f^3} \left(\frac{3(g' - g''')ii}{2(i-m)^2} - \frac{(3g' - g''')ii}{(i-m)(i-2m)} - \frac{(g' - 3g''')ii}{(i-m)(3i-2m)} \right) \end{array} \right\}$$

$D' = 0$; $E' = 0$; $F' = 0$; & $G' = 0$.

Hanc ob rem tota correctio ita contrahitur, ut tantum duobus terminis conſtet; ſitque $\varphi = \zeta \pm Æ + n\, B' \text{ſin.}\, \eta + n\, C' \text{ſin.}\, 2\eta$; ſiquidem in perturbationibus excentricitatem k rejicimus.

§. CXXIV. Diſtantia vera planetæ à ſole x nunc quoque facile definiri poterit; cum enim ſit $x = \frac{p}{1 - q \cos. \nu}$ ſi ponamus ut ſupra $p = b(1 + nP)$ & $q = k + nQ$, erit ob nP & nQ minima:

$$x = \frac{b}{1 - k \cos. \nu} + nb(P + Q \cos. \nu).$$

Supra autem jam valores quantitatum P & Q aſſignavimus, hic vero pro ν capi debet ea anomalia vera, quæ

anomaliæ mediæ correctæ ϑ' reſpondet: ſin autem anomalia vera tabulari uti velimus, eamque littera ν indicemus, pro ν in iſta formula ſcribere debemus

$$\nu - \frac{n}{k} \Big(\alpha \text{ ſin.}\, \nu + \beta \text{ ſin.}\, (\eta - \nu) + \gamma \text{ ſin.}\, (\eta + \nu) + \delta \text{ ſin.}\, (2\eta - \nu) + \varepsilon \text{ ſin.}\, (2\eta + \nu) \Big);$$

ideoque pro $k \cos. \nu$ ſcribi opportebit:

$$k \cos. \nu + \frac{n}{2} \Big(\alpha - \alpha \cos. 2\nu - (\beta - \gamma) \cos. \eta + \beta \cos. (\eta - 2\nu) + \gamma \cos. (\eta + 2\nu) - (\delta - \varepsilon) \cos. 2\eta + \delta \cos. (2\eta - 2\nu) - \varepsilon \cos. (2\eta + 2\nu) \Big).$$

Hinc ſi ponamus $k \cos. \nu = k \cos. \nu + nR$, erit $x = \frac{b}{1 - k \cos. \nu} + nb(P + Q \cos. \nu + R)$, ubi $\frac{b}{1 - k \cos. \nu}$ diſtantiam ex tabulis more ſolito erutam exprimit; neque vero plerumque operæ eſt pretium pro diſtantia hanc correctionem adhibere.

SECTIO IX.

Evolutio Inæqualitatum quibus cum linea nodorum tum inclinatio ab actione Planetarum afficitur.

§. CXXV. SUPRA in §. LXXXVII & LXXXVIII formulas exhibuimus differentiales, quibus mutatio momentanea tam in situ lineæ nodorum quam in inclinatione orbitæ planetæ perturbati ad orbitam perturbantis, quam tanquam fixam considero exprimitur. Productæ autem sunt istæ formulæ usque ad terminos excentricitate simplici k affectos, omissis iis, qui vel per quadratum altioremve potestatem ejusdem excentricitatis k, vel per excentricitatem orbitæ planetæ perturbantis e sunt multiplicati, quos autem si quis laborem suscipere velit eidem methodo insistendo non esset difficile insuper adjicere: neque etiam tum istarum formularum integratio majori premeretur difficultate. Verum quia actio planetarum est minima, hic adeo terminos excentricitatem k involventes rejicere licebit, sicque expressiones integrales & facilius invenientur, & multo fient simpliciores.

§. CXXVI. Quod igitur primum ad longitudinem nodi attinet, quam respectu stellarum fixarum littera π indicavimus, in ejus differentiale ingreditur angulus σ, qui denotat argumentum latitudinis $\varphi - \pi$. Cum ergo hoc calculo negligamus, & differentiale ipsius $d\pi$ præ $d\varphi$ sit minimum, tuto assumere licet $d\sigma = d\varphi = i\,d\omega$, & quia porro est $d\eta = (i-m)\,d\omega$,

integrando obtinebimus pro longitudine lineæ nodorum:

$$\pi = Const. - \frac{nbbc}{4f^3}\cdot g' i\omega + \frac{nibb}{cc}\left(\frac{\sin.\eta}{2(i-m)} + \frac{\sin.(\eta-2\sigma)}{2(i+m)}\right)$$
$$-\frac{nibbc}{4f^3}\left\{\begin{aligned}&\frac{(2g+g'')\sin.\eta}{i-m} + \frac{(g'+g''')\sin.2\eta}{2(i-m)} + \frac{2g\sin.(\eta-2\sigma)}{i+m}\\ &-\frac{g'\sin.2\sigma}{2i} + \frac{g'\sin.(2\eta-2\sigma)}{2m} - \frac{g''\sin.(\eta+2\sigma)}{3i-m}\end{aligned}\right\}$$

Hinc ergo erit longitudo media nodi $= Const. - \frac{nbbc}{4f^3}\cdot g' i\omega$, & quia g' semper est quantitas positiva, patet lineam nodorum semper regredi, & quidem singulis annis per angulum $= \frac{90^\circ\, nbbc}{f^3}\cdot ig'$ graduum, ponendo $= 360^\circ$.

§. CXXVII. Formulam pro differentiali $\frac{d.\,tang.\,G}{tang.\,G}$ inventam, quia etiam est valde parva, loco $tang.\,G$ poterimus per tangentem inclinationis mediæ multiplicare, sit igitur inclinatio media $= \lambda$, denotante G inclinationem veram, atque integrando obtinebimus

$$\frac{tang.\,G}{tang.\,\lambda} = 1 + \frac{nibb}{2cc}\left(\frac{\cos.\eta}{i-m} - \frac{\cos.(\eta-2\sigma)}{2+m}\right)$$
$$+\frac{nibbc}{4f^3}\left\{\begin{aligned}&\frac{(2g-g'')\cos.\eta}{i-m} + \frac{(g'-g''')\cos.2\eta}{2(i-m)} - \frac{2g\cos.(\eta-2\sigma)}{i+m}\\ &+\frac{g'\cos.2\sigma}{2i} - \frac{g'\cos.(2\eta-2\sigma)}{2m} - \frac{g''\cos.(\eta+2\sigma)}{3i-m}\end{aligned}\right\}$$

Cum igitur inclinatio vera G minime discrepet à media λ, ponamus $G = \lambda + d\lambda$, eritque $tang.\,G = tang.\,\lambda + \frac{d\lambda}{\cos.\lambda^2}$, & $\frac{tang.\,G}{tang.\,\lambda} = 1 + \frac{d\lambda}{\sin.\lambda\cos.\lambda} = 1 + \frac{2d\lambda}{\sin 2\lambda}$; qua formula cum illa expressione collata eliciemus valorem ipsius $d\lambda$, quo substituto reperietur

$$G = \lambda + \frac{nibb\, fin.\, 2\lambda}{4cc}\left(\frac{cof.\,\nu}{i-m} - \frac{cof.\,(\nu - 2\sigma)}{i+m}\right)$$

$$-\frac{nibbc\, fin.\, 2\lambda}{8f^3}\left\{\begin{array}{l} +\frac{(2g - g'')\, cof.\,\nu}{i-m} - \frac{2g\, cof.\,(\nu - 2\sigma)}{i+m} - \frac{g'\, cof.\,(2\nu - 2\sigma)}{2m} \\ +\frac{(g' - g''')\, cof.\, 2\nu}{2(i-m)} - \frac{g'\, cof.\, 2\sigma}{2i} - \frac{g''\, cof.\,(\nu + 2\sigma)}{3i - m} \end{array}\right\}$$

§. CXXVIII. Inæqualitates igitur iſtæ non ſolum ob fractionem minimam n ſed etiam ob *ſin.* 2λ erunt tam exiguæ, ut nullo modo obſervari queant: atque etiam inæqualitates periodicæ in linea nodorum vix unquam in ſenſus occurrant, unde in uſu aſtronomico tuto negligi poterunt. Tantum ergo notaſſe ſufficiet motum lineæ nodorum medium, qui continetur hac formula:

$$\pi = Conſt. - \frac{nbbc}{4f^3} g' i\omega;$$

unde conſtat lineam nodorum motu uniformi contra ſignorum ſeriem recedere. Etſi enim hic motus ſingulis annis ſit tardiſſimus ut percipi nequeat, tamen ſucceſſione plurium annorum, quia continuo accumulatur, maxime ſenſibilis evadere poteſt. Effectus autem qui inde in phænomena Aſtronomica redundat, in hoc potiſſimum cernetur, quod ſi pro planeta perturbato terra accipiatur, latitudo ſtellarum fixarum aliquantillum immutetur, qui effectus propterea imprimis meretur, ut accuratius evolvatur.

§. CXXIX. Iſta autem latitudinis mutatio pendebit à longitudine cujuſque ſtellæ fixæ ratione nodorum: Poſita enim longitudine nodi aſcendentis terræ ſuper orbita planetæ perturbantis $= \pi$, quæ convenit cum longitudine nodi deſcendentis ejuſdem planetæ ſuper ecliptica, ſi longitudo cujuſpiam ſtellæ fixæ fuerit $= \pi$, ejus latitudo ſi fuerit borealis poſt tempus, quo ſol arcum ω abſolvit, diminuetur particula $= \frac{nbbc}{4f^3} g' i\omega\, ſin.\,\lambda$, ſin autem latitudo fuerit auſtralis tantumdem augebitur. Contrarium eveniet ſi longitudo ſtellæ fuerit $180° + \pi$, tum enim eodem tempore, cui ſolis motus ω reſpondet, ejus latitudo ſi fuerit borealis augebitur particula $\frac{nbbc}{4f^3} g'.\, i\omega\, ſin.\,\lambda$, ſin autem ſit auſtralis tantumdem diminuetur. At ſi longitudo ſtellæ 90° diſtet à nodis tum ejus latitudo nullam patietur mutationem. In genere autem ſi longitudo ſtellæ fixæ fuerit $= \xi$, eodem tempore ejus latitudo ſi fuerit borealis diminuetur particula $= \frac{nbbc}{4f^3} g' i\omega\, ſin.\,\lambda\, coſ.\,(\xi - \pi)$ ſin autrem latitudo ſit auſtralis tantumdem augebitur.

§. CXXX. Maxime igitur notabiles effectus, qui ab actione planetarum in terram exercentur, ſunt primo iſta exigua mutatio in latitudine ſtellarum fixarum, quæ autem cum obſervationibus vetuſtis circa latitudinem ſtellarum fixarum inſtitutis minus fidere liceat, utrum veritati ſit conformis? non tam facile explorari poteſt. Interim tamen ſtudioſa collatio veterum obſervationum cum recentioribus vix dubitare ſinit, quin in quibus ſtellis fixis latitudo parumper ſit immutata, quod phænomenum ſine dubio actioni planetarum eſt tribuendum. Deinde maxime conſpicuus effectus cernitur in motu aphelii, cujus conſenſus cum veritate facillime explorari poteſt, quandoquidem ex obſervationibus certum eſt, aphelium terræ quotannis per ſpatiolum 11″ circiter promoveri; ſimilique modo motus apheliorum in reliquis planetis ab eorum actione mutua oriundus cum obſervationibus comparari poterit. Reliqui effectus in pleriſque planetis minus perceptibiles conſiſtunt in mutatione excentricitatis, in inæqualitatibus periodicis loci apheliorum, unde anomalia media afficitur ac denique in variatione parametri orbitarum; quibus cognitis, loca planetarum per præcepta vulgaria Aſtronomica facile aſſignari poterunt.

PARS ALTERA

Continens Applicationem Theoriæ ad motum Terræ ejuſque perturbationes ab actione reliquorum Planetarum oriundas.

1. IN parte ſuperiori Theoriam actionis planetarum mutuæ ita in genere conſtitui, ut ex ea inæqualitates cujuſque planetæ, quæ ejus motui ab actione reliquorum planetarum inducuntur, definiri atque aſſignari queant. Quas inæqualitates ita ad commodum calculi aſtronomici traduxi, ut pateat, quantum primo latus rectum ſeu parameter orbitæ, tum vero excentricitas, tertio locus aphelii, & quarto poſitio plani, quod orbita in cœlo occupat, quovis tempore immutetur. Cognitis enim his variationibus, manifeſto apparebit, quantum motus planetæ quovis tempore à regulis Keplerianis recedere, & quales correctiones Tabulis conſuetis adhiberi debeant, ut ad quodvis tempus verus planetæ locus in cœlo aſſignari queat.

2. Labor autem foret nimis operoſus, limiteſque huic diſſertationi præfixos longe excederet, ſi hanc Theoriam ad ſingulos planetas accommodare vellem. Ipſa quoque Illuſtriſſima Academia Regia tam prolixum opus non requirit, dum poſtquam Theoria perturbationum ſolide fuerit ſtabilita ejus applicationem tantum ad motum Terræ exigit: cujus præcepto morem geſturus

cunctas perturbationes, quibus terra in motu ſuo ob actionem reliquorum planetarum eſt obnoxia, data opera determinabo. Ex hac autem applicatione facile perſpicietur, quomodo per eandem Theoriam & reliquorum planetarum omnium perturbationes, quas ſibi mutuo induunt, definiri oporteat.

3. Ad motum autem terræ perturbandum reliqui planetæ omnes concurrunt, ſingulorumque effectus ſecundum præcepta ſuperiora ſeorſim inveſtigari conveniet, quod opus pro ſingulis ſimili calculo abſolvetur. Quoniam igitur terram in locum planetæ perturbati conſtituimus, littera i perpetuo unitatem denotabit: atque ex tabulis ſolaribus pro ejus excentricitate media aſſumemus $k = 0,0168$. Quanquam enim cunctis inæqualitatibus rite determinatis demum verum valorem excentricitatis mediæ k definire licet, tamen in ipſa harum inæqualitatum inveſtigatione valore ipſius k proxime vero tuto uti poterimus, quandoquidem hic minimas aberrationes merito negligimus. Interim valor $k=0,0168$ tam prope ad veritatem accedere videtur, ut error nullius certe ſit momenti. Habebimus igitur conſtanter $i = 1$ & $k = 0,0168$, neque quicquam præterea ex terræ theoria repeti eſt neceſſe, propterea quod non tam quantitas abſoluta ejus parametri quam ejus ratio ad parametrum cujuſque alterius planetæ in computum ingreditur.

4. Quicunque planetarum pro perturbante aſſumitur, ejus primum vim abſolutam, ſeu rationem ejus maſſæ ad maſſam Solis noſſe opportet, quam rationem littera n indicavimus. Ex phænomenis quidem Satellitum Newtonus concluſit, ſi Saturnus ſit planeta perturbans fore $n = \frac{1}{3021}$, ſin autem ſit Jupiter eſſe $n = \frac{1}{1067}$; pro reliquis autem planetis, quoniam Satellitibus deſtituuntur, valor fractionis n ex phænomenis determinari nequit. Etſi autem Mars & Venus ratione voluminis terræ ſunt minores, fortaſſe ob majorem denſitatem ratione maſſæ non multum diſcrepant, foretque ergo pro illis $n = \frac{1}{170000}$; pro Marte tamen hanc fractionem ob celeb. Monnierii obſervationes notabiliter imminuere vellem, ut eſſet quaſi $n = \frac{1}{2000000}$, nullumque eſt dubium, quin pro Mercurio hæc fractio multo minor ſit accipienda forſitan $n = \frac{1}{10000000}$. Verum ex ipſa quantitate effectuum forte hæc accuratius definire licebit.

5. Porro pro quovis planeta noſſe oportet motum medium ſeu rationem anguli, dato tempore circa ſolem deſcripti ad motum medium ſolis pro eodem tempore. Hanc rationem littera m indicavimus, unde tabulas aſtronomicas conſulentes reperiemus

pro Saturno	$m = \frac{2}{59} = 0,0339$
pro Jove	$m = \frac{7}{83} = 0,0843$
pro Marte	$m = \frac{42}{79} = 0,5316$
pro Venere	$m = \frac{13}{8} = 1,6250$
pro Mercurio	$m = \frac{137}{33} = 4,1515$

Excentricitate horum planetarum littera e indicata non erit opus, ſiquidem vidimus perturbationes inde pendentes tam prodire exiguas, ut præ reliquis facile rejici queant. Saltem in hac applicatione ejus rationem non habebimus, etiamſi in Theoria non ſit neglecta; propterea quod ad motum apogei medium nihil plane confert.

6. Cum igitur ſit $m\frac{a\sqrt{a}}{c\sqrt{c}} = \frac{b\sqrt{b}}{c\sqrt{c}}$ ob $a = b$, hinc reliquas expreſſiones, quæ in calculum ingrediuntur, determinare poterimus.

Sic erit $\frac{bb}{cc} = \sqrt[3]{m^4}$; $\frac{bb}{ff} = \frac{\sqrt[3]{m^4}}{1+\sqrt[3]{m^4}}$; $\frac{cc}{ff} = \frac{1}{1+\sqrt[3]{m^4}}$;

hincque $\frac{b^3}{f^3} = \frac{mm}{(1+\sqrt[3]{m^4})^{\frac{3}{2}}}$; $\frac{bbc}{f^3} = \frac{\sqrt[3]{m^4}}{(1+\sqrt[3]{m^4})^{\frac{3}{2}}}$.

Tum etiam hinc elicientur valores numerorum μ & ν supra (§. LXXVII) introductorum, eritque:

$$\mu = \frac{2bc}{ff} = \frac{2\sqrt[3]{mm}}{1+\sqrt[3]{m^4}}; \ \&$$

$$\nu = \frac{cc - bb}{ff} = \frac{1-\sqrt[3]{m^4}}{1+\sqrt[3]{m^4}}.$$

Ex his autem neglecta excentricitate e habemus:

$$\frac{1}{r^3} = \frac{1}{f^3}\left(1 - \tfrac{3}{2}(1-\nu)\, k\, cos.\, v\right) = \frac{1}{f^3}\left(1 - \frac{3\sqrt[3]{m^4}}{1+\sqrt[3]{m^4}} k\, cos.\, v\right)$$

$$s = \mu\,(1 + \nu\, k\, cos.\, v);\quad ss = \mu^2\,(1 + 2\nu\, k\, cos.\, v)$$

$$s^3 = \mu^3\,(1 + 3\nu\, k\, cos.\, v) \ \&c.$$

atque $1 - ss = \nu\nu - 2\mu\mu\nu\, k\, cos.\, v$, &

$$\frac{1}{1-ss} = \frac{1}{\nu^2} + \frac{2\mu\mu\, k\, cos.\, v}{\nu^3}.$$

7. Jam præcipuus labor in computo litterarum g, h, g', h', &c. consistet, pro quibus primum ex §. LXXII valores expressionum

$$P(1-ss);\ \tfrac{1}{2}Q(1-ss);\ \tfrac{1}{2}R(1-ss);\ \tfrac{1}{2}S(1-ss)\ \&c.$$

hincque ipsæ hæ litteræ P, Q, R, S, &c. colligi debent. Quæ singulæ cum habituræ sint formam $A + B\, k\, cos.\, v$, erit porro

$$g + h\ k\, cos.\, v = P\ \left(1 - \tfrac{3}{2}(1-\nu)\, k\, cos.\, v\right)$$
$$g' + h'\ k\, cos.\, v = Qs\ \left(1 - \tfrac{3}{2}(1-\nu)\, k\, cos.\, v\right)$$
$$g'' + h''\ k\, cos.\, v = Rs^2\ \left(1 - \tfrac{3}{2}(1-\nu)\, k\, cos.\, v\right)$$
$$g''' + h'''\ k\, cos.\, v = Ss^3\ \left(1 - \tfrac{3}{2}(1-\nu)\, k\, cos.\, v\right)$$
&c.

quoniam excentricitatem e, ac proinde numeros inde pendentes l, l', l'', &c. negligimus. Negligimus vero etiam terminos quadratum k^2 ejusque altiores potestates involventes, unde calculus in numeris satis expedite absolvi poterit. Atque hoc modo omnia elementa, quæ ad perturbationes motus in orbita inveniendas spectant, erunt cognita.

8. Denique vero quod ad variationem plani orbitæ attinet id pro quovis planeta perturbante ad planum ejus orbitæ, quæ saltem ad tempus ut fixa spectatur, est relatum. Ex tabulis autem Astronomicis colligimus pro An. 1750.

Si orbita terræ referatur ad	Esse longitudinem nodi ascendentis	Inclinationem orbitæ
Orbitam Saturni	9^s, 21°, $20'$, $6''$	2°, $30'$, $10''$
Orbitam Jovis	9, 8, 15, 50,	1, 19, 10
Orbitam Martis	7, 17, 56, 22,	1, 51, 0
Orbitam Veneris	8, 14, 23, 43,	3, 23, 20
Orbitam Mercurii	7, 18, 29, 0,	6, 59, 20

His igitur notatis perturbationes, quas quilibet planeta in motu terra producit, per calculum numericum investigemus, unde quantum Theoria cum veritate consentiat, facile erit judicare.

I.

Investigatio inæqualitatum motus Terræ ab actione Saturni oriundarum.

9. PRIMUM igitur Saturnus locum teneat planetæ perturbantis, atque ut vidimus pro eo habemus

$$n = \tfrac{1}{3021} \ \& \ m = 0{,}0339,$$

& quantitates hinc derivatas cum suis logarithmis:

$$\tfrac{bb}{cc} = 0{,}01097 \qquad l\tfrac{bb}{cc} = 8{,}040266$$
$$\tfrac{b^3}{f^3} = 0{,}00114 \qquad l\tfrac{b^3}{f^3} = 7{,}055293$$
$$\tfrac{bbc}{f^3} = 0{,}01079 \qquad l\tfrac{bbc}{f^3} = 8{,}033159$$

$$\mu = 0{,}2072; \quad l\mu = 9{,}316425; \quad \mu\nu = 0{,}2027$$
$$\mu^2 = 0{,}0429; \quad l\mu^2 = 8{,}632850; \quad \mu^2\nu = 0{,}0420$$
$$\mu^3 = 0{,}0089; \quad l\mu^3 = 7{,}949275; \quad \mu^3\nu = 0{,}0087$$
$$\mu^4 = 0{,}0018; \quad l\mu^4 = 7{,}265700; \quad \mu^4\nu = 0{,}0018$$
$$\nu = 0{,}9783; \quad l\nu = 9{,}990465; \quad 1-\nu = 0{,}0217$$
$$\tfrac{1}{\nu\nu} = 1{,}0449; \quad l\tfrac{1}{\nu\nu} = 0{,}019070; \quad \tfrac{\mu\mu}{\nu^3} = 0{,}0459$$

10. Ex his valoribus formabimus sequentes

$$\tfrac{f^3}{r^3} = 1 - \tfrac{3}{2}(1-\nu)\,k\,cos.\,v = 1 - 0{,}0326\,k\,cos.\,v$$
$$s = 0{,}2072 + 0{,}2027\,k\,cos.\,v$$
$$s^2 = 0{,}0429 + 0{,}0840\,k\,cos.\,v$$
$$s^3 = 0{,}0089 + 0{,}0261\,k\,cos.\,v$$
$$s^4 = 0{,}0018 + 0{,}0071\,k\,cos.\,v$$
$$s^5 = 0{,}0003 + 0{,}0017\,k\,cos.\,v$$
$$s^6 = 0{,}0001 + 0{,}0003\,k\,cos.\,v$$

Tum

Tum vero $\tfrac{1}{1-\nu\nu} = 1{,}0449 + 0{,}0918\,k\,cos.\,v$, unde valores $P(1-ss)$; $\tfrac{1}{2}Q(1-ss)$; $\tfrac{1}{2}R(1-ss)$, &c. colliguntur

$$P(1-ss) = 0{,}99729 - 0{,}00535\,k\,cos.\,v$$
$$\tfrac{1}{2}Q(1-ss) = 0{,}75307 + 0{,}00609\,k\,cos.\,v$$
$$\tfrac{1}{2}R(1-ss) = 0{,}47518 + 0{,}01283\,k\,cos.\,v$$
$$\tfrac{1}{2}S(1-ss) = 0{,}28004 + 0{,}01326\,k\,cos.\,v$$

11. Ex his deducitur:

$$P = 1{,}0421 + 0{,}0859\,k\,cos.\,v; \quad \text{hinc}$$
$$Q = 1{,}5746 + 0{,}1512\,k\,cos.\,v; \quad Qs = 0{,}3257 + 0{,}3500\,k\,cos.\,v$$
$$R = 0{,}9930 + 0{,}1140\,k\,cos.\,v; \quad Rs^2 = 0{,}0425 + 0{,}0877\,k\,cos.\,v$$
$$S = 0{,}5852 + 0{,}0790\,k\,cos.\,v; \quad Ss^3 = 0{,}0052 + 0{,}0161\,k\,cos.\,v$$

multiplicentur jam hæ formulæ per $1 - 0{,}0326\,k\,cos.\,v$, indeque pro litteris g, h, g', h', g'', h'', &c. sequentes obtinebuntur valores:

$$g = 1{,}0421; \qquad h = 0{,}0519$$
$$g' = 0{,}3257; \qquad h' = 0{,}3394$$
$$g'' = 0{,}0425; \qquad h'' = 0{,}0863$$
$$g''' = 0{,}0052; \qquad h''' = 0{,}0159$$

&c.

qui per se tantopere decrescunt, ut circa convergentiam seriei in quam supra terminum $\tfrac{1}{z^3}$ transformavimus nullum dubium superesse possit.

12. His valoribus inventis inquiramus primo in motum aphelii terræ, quatenus ab actione Saturni afficitur, & quoniam per (101) tempore quo Sol motu medio conficit angulum ω, aphelium terræ respectu stellarum fixarum promovetur per spatiolum

$$\tfrac{nbbc}{4f^3}(2g' + h')\,\omega - \tfrac{nb^3}{2f^3}(3g + h)\,\omega, \text{ ob}$$

$\frac{bbc}{4f^3} = 0,002698; \quad 2g' + h' = 0,9908$, erit

$\frac{bbc}{4f^3}(2g' + h') = 0,002673;$

$\frac{b^3}{2f^3} = 0,000570; \quad 3g + h = 3,1782$, erit

$\frac{b^3}{2f^3}(3g + h) = 0,001811.$

Hinc isto tempore aphelium proferetur per spatium

$$0,000862\, n\omega = \frac{0,000862\,\omega}{3021}, \text{ ob } n = \frac{1}{3021}.$$

Tempore ergo unius anni, quo $\omega = 360° = 1296000''$, aphelium terræ à Saturno propellitur per spatium $= 0,370'' = 22'''$, ideoque tempore 100 annorum per spatium $= 37''$, si ergo terra tantum à Saturno perturbaretur, aphelium respectu stellarum fixarum promoveretur :

Tempore unius anni per spatium $22'''$,
Tempore 100 annorum per spatium $37''$.

13. Hinc ad quodvis tempus longitudo media aphelii terræ innotescit, quæ autem porro per inæqualitates periodicas corrigi debet. Pendent autem eæ à duobus angulis η & ν, quorum ille η habetur si longitudo Saturni θ à longitudine terræ φ subtrahatur, hic vero ν denotat anomaliam terræ veram. Cum igitur sit $\frac{bb}{2cc} = 0,005485$; & $m = 0,0339$, hinc $2i - m = 1,9661$; $i - 2m = 0,9322$, & $3i - 2m = 2,9322$, ob $n = \frac{1}{3021}$ & $k\ 0,0168$, formula pro motu aphelii (§. CI) inventa ad angulos reducta dabit :

Longitudo Aphelii vera = Longitudini aphelii mediæ

$$+ 1973''\, \text{fin.}\,(\eta - \nu) + 11''\, \text{fin.}\,(\eta + \nu)$$
$$- 5''\, \text{fin.}\,\nu + 22\, \text{fin.}\,(\eta - \nu) - \tfrac{1}{2}\, \text{fin.}\,(\eta + \nu)$$

$$+ 7\, \text{fin.}\,\nu - 2010\, \text{fin.}\,(\eta - \nu) - 11\, \text{fin.}\,(\eta + \nu)$$
$$+ 12''\, \text{fin.}\,(2\eta - \nu) - 1''\, \text{fin.}\,(2\eta + \nu);$$

unde patet has inæqualitates tantum non se mutuo destruere dum eæ reducuntur ad

$$+ 2''\, \text{fin.}\,\nu - 15''\, \text{fin.}\,(\eta - \nu) + 12''\, \text{fin.}\,(2\eta - \nu),$$

quare dum nunquam ad dimidium minutum assurgunt, tuto negligi possunt; ita ut sufficiat effectum in motu aphelii medio notasse.

14. Variationes, quæ ab actione Saturni excentricitati & parametro inducuntur, tam sunt exiguæ ut omnino sentiri nequeant. Neque vero etiam has inæqualitates evolvisse est opus, cum quoniam sunt minimæ, supra (§. CXXIII) effectum inde in locum terræ redundantem expresserimus, sumta scilicet æquatione, quæ secundum tabulas ordinarias anomaliæ mediæ convenit, quæ sit $= \pm Æ$, & posita longitudine terræ media $= \zeta$, vidimus fore longitudinem ejus veram

$$\varphi = \zeta \pm Æ + n B'\, \text{fin.}\,\eta + n C'\, \text{fin.}\,2\eta.$$

Ibidem autem valores litterarum B' & C' dedimus, ex quibus hos coëfficientes in minutis secundis colligimus;

$$n B' = -\tfrac{1}{2}'', \ \& \ n C' = 0;$$

unde patet longitudinem terræ regulis ordinariis computatam nullam sensibilem alterationem ab actione Saturni pati, cum ea vix dimidio minuto secundo mutari possit. Pro orbita igitur terræ nil aliud relinquitur, nisi exigua illa aphelii terræ promotio, cujus effectus post integrum seculum demum ad $37''$ exsurgit.

15. Tantum ergo superest, ut in mutationem plani, in quo orbita terræ versatur, inquiramus; à Saturno autem linea nodorum, seu intersectio orbitarum terræ & Saturni contra signorum ordinem removebitur tem-

pore quo sol motu medio angulum ω absolvit per spatiolum $= \frac{nbbc}{4f^3} \cdot g' \omega = \frac{\omega}{3438000}$. Hinc ergo singulis annis linea nodorum super orbita Saturni regredietur per $0, 377''$, seu $22'''$, sæculo autem elapso hic motus erit quasi $38''$. Inæqualitates periodicas, quibus locus nodi afficitur, quia nullius plane sunt momenti, hic non evolvo, multoque minus eas, quibus in genere inclinatio turbari est inventa: illæ enim nunquam ad minutum secundum, hæ vero ne ad tertium quidem assurgere reperientur. Si ergo terra à solo Saturno perturbantur, linea nodorum terræ super orbita Saturni retromoveretur

Tempore unius anni per spatium $22'''$,

Tempore unius seculi per spatium $38''$,

qui ergo motus motui aphelii proxime est æqualis.

16. Phænomena, quæ hinc in latitudinem stellarum fixarum fluunt, ita se habebunt. Cum sit inclinatio orbitæ terræ ad orbitam Saturni $\lambda = 2^\circ, 30', 10''$, erit pro tempore unius anni $\frac{nbbc}{4f^3} g' . \omega \, fin. \lambda = 0, 0164''$: Hinc stellarum fixarum, quarum longitudo est $9^s, 21^\circ$, vel $3^s, 21^\circ$, latitudo tempore unius anni mutabitur fere uno minuto tertio. Seculo autem elapso, mutatio latitudinis ita se habebit:

Si longitudo stellæ sit $9^s, 21^\circ$ circiter

ejus latitudo $\left\{ \begin{matrix} \text{decrescit} \\ \text{crescit} \end{matrix} \right\}$ $1'', 38'''$ si latitudo fuerit $\left\{ \begin{matrix} \text{borealis} \\ \text{australis} \end{matrix} \right\}$

At si longitudo stellæ sit $3^s, 21^\circ$ circiter

ejus latitudo $\left\{ \begin{matrix} \text{crescit} \\ \text{decrescit} \end{matrix} \right\}$ $1'', 38'''$ si latitudo fuerit $\left\{ \begin{matrix} \text{borealis} \\ \text{australis} \end{matrix} \right\}$

Hujusmodi ergo stellarum fixarum latitudo intervallo decem seculorum mutari potuit $16'', 24'''$, idque ob solam actionem Saturni.

II.

Investigatio inæqualitatum Terræ ab actione Jovis oriundarum.

17. COLLOCATO jam Jove in locum planetæ perturbantis habebimus

$$n = \tfrac{1}{1067}, \ \& \ m = 0, 0843;$$

indeque quantitates derivatas cum logarithmis subscriptis

$\frac{bb}{cc} = 0, 0369643$; $\frac{b^3}{f^3} = 0, 0067303$; $\frac{bbc}{f^3} = 0, 0350044$
8, 567770; 7, 828010; 8, 544124

Porro erit $\mu = 0, 37081$ & $\nu = 0, 97871$, hincque

$s = 0, 37081 + 0, 34437 \, k \, cof. \nu$;
9, 569151 9, 537029

$s^2 = 0, 13750 + 0, 25539 \, k \, cof. \nu$;
9, 138302 9, 407210

$s^3 = 0, 05099 + 0, 14206 \, k \, cof. \nu$;
8, 707453 9, 152452

$s^4 = 0, 01891 + 0, 07023 \, k \, cof. \nu$;
8, 276604 8, 846542

$s^5 = 0, 00701 + 0, 03255 \, k \, cof. \nu$;
7, 845755 8, 512603

$s^6 = 0, 00260 + 0, 01449 \, k \, cof. \nu$;
7, 414906 8, 160935

$s^7 = 0, 00096 + 0, 00627 \, k \, cof. \nu$;
6, 985057 7, 797033

$$\begin{array}{ll} s^8 = 0,00036 & + 0,00266\ k\ cos.\ v; \\ \quad 6,553208 & \quad 7,424176 \\ s^9 = 0,00013 & + 0,00111\ k\ cos.\ v; \\ \quad 6,122359 & \quad 7,044479 \\ s^{10} = 0,00005 & + 0,00046\ k\ cos.\ v; \\ \quad 5,691510 & \quad 6,659388 \\ \frac{1}{1-ss} = 1,15943 & + 0,34332\ k\ cos.\ v; \\ \quad 0,064244 & \quad 9,535698 \\ \frac{f^3}{r^3} = 1 - 0,10694 & k\ cos.\ v; \\ \quad 9,029140 & \end{array}$$

18. Ex his jam calculo ſecundum (§. LXXII) ſubducto invenitur

$$\begin{array}{ll} P\ (1-ss) = 0,991111 & - 0,017094\ k\ cos.\ v; \\ \quad 9,996122 & \quad 8,232844 \\ Qs(1-ss) = 0,563774 & + 0,538394\ k\ cos.\ v; \\ \quad 9,751105 & \quad 9,731100 \\ Rs^2(1-ss) = 0,134856 & + 0,262368\ k\ cos.\ v; \\ \quad 9,129870 & \quad 9,418910 \\ Ss^3(1-ss) = 0,030176 & + 0,088736\ k\ cos.\ v; \\ \quad 8,479518 & \quad 8,948100 \end{array}$$

haſque formulas primum per $\frac{1}{1-ss}$ deinde per $\frac{f^3}{r^3}$ multiplicari oportet, hoc eſt conjunctim per

$$\begin{array}{ll} 1,15943 & + 0,21933\ k\ cos.\ v; \\ 0,064244 & \quad 9,341098 \end{array}$$

unde prodeant formæ $g + h\ k\ cos.\ v$. Facta igitur multiplicatione reperietur:

$$\begin{array}{ll} g = 1,14912; & h = 0,19756; \\ g' = 0,65366; & h' = 0,74788; \\ g'' = 0,15636; & h'' = 0,33378; \\ g''' = 0,03498; & h''' = 0,10950; \end{array}$$

19. Hinc pro motu aphelii terræ medio erit

$$2g' + h' = 2,05520; \quad 3g + h = 3,64492$$
$$\frac{bbc}{4f^3}(2g' + h') = 0,017985; \quad \frac{b^3}{2f^3}(3g + h) = 0,012265;$$

unde tempore, quo ſol motu medio angulum ω conficit, aphelium terræ promovetur per ſpatium

$$0,005720\ n\omega = \frac{\omega}{186538}, \text{ ob } n = \frac{1}{1067}.$$

Ponamus jam $\omega = 360^\circ = 1296000''$, ut obtineamus aphelii motum annuum, qui prodibit $= 6'', 95 = 6'', 57'''$; & motus ſecularis $= 695'' = 11', 35''$. Quare ſi terra tantum à Jove perturbaretur, aphelium ejus reſpectu ſtellarum fixarum promoveretur

Tempore unius anni per ſpatium $6'', 57'''$,

Tempore centum annorum per ſpatium $11', 35''$,

Saturnus igitur & Jupiter conjunctim imprimunt aphelio terræ motum annuum $7'', 19'''$. Revera autem quotannis promoveri obſervatur per ſpatium $11''$ circiter.

20. Omiſſis mutationibus, quæ excentricitatem & parametrum afficiunt, quæramus ſtatim correctionem, quam locus terræ in orbita exigit, ac pro coëfficientibus B' & C' (§. CXXIII) obtinemus valores ſequentes:

$$B' = -0,03650; \quad C' = +0,01381.$$

Quare ſi η jam denotet angulum, qui oritur, ſi longitudo Jovis à longitudine terræ ſubtrahatur, longitudo terræ per tabulas ſolito more computatas ſequentem correctionem recipere debet:

$$-7'', 06\ \mathit{ſin.}\ \eta + 2'', 67\ \mathit{ſin.}\ 2\eta;$$

quæ ergo nunquam ad decem minuta ſecunda exſurgere poteſt. Verumtamen hæ correctiones maximi ſunt momenti, quandoquidem Theoria motus ſolis jam ad tantam perfectionem eſt perducta, ut in calculo vix unum minutum ſecundum negligere fas ſit. Deinde cum Luna fore tantumdem motum terræ perturbare ſit inventa, neuter effectus per obſervationes rite comprobari poteſt, niſi utriuſque vera quantitas per Theoriam ſit explorata.

21. Motus lineæ nodorum motui aphelii tam exacte æqualis deprehenditur, ut differentia vix ad partes millioneſimas aſcendat, diſcrimen autem in hoc verſatur, quod aphelium ſecundum ſignorum ordinem progreditur, dum nodi motum retrogradum tenent. Motus igitur hujus lineæ nodorum ita eſt comparatus ut retrogrediatur

Tempore unius anni per ſpatium 6'', 57''',

Tempore 100 annorum per ſpatium 11', 35''.

Viciſſim ergo linea nodorum orbitæ Jovis ad eclipticam relatæ tanto motu retrogredietur, quatenus ipſa terra actioni Jovis eſt ſubjecta: qui effectus probe diſtingui debet ab eo, quem reliqui planetæ actione ſua immediate in Jovem exerunt, unde peculiaris lineæ nodorum Jovis motus oritur non pendens à mobilitate plani eclipticæ. Ex quo intelligitur motum obſervatum nodorum cujuſque planetæ eſſe effectum mixtum partim ex mobilitate ejus propriæ orbitæ partim vero ex mobilitate ipſius eclipticæ oriundum, qui propterea modo magis rationalis definietur, ſi orbitæ planetarum non cum plano eclipticæ, utpote mobili, verum cum plano reſpectu ſtellarum fixo, veluti forſitan cum plano æquatoris ſolis comparentur.

22. Seorſim autem hæc mobilitas orbitæ terræ ab actione Jovis profecta ſentiri debet in latitudine ſtellarum fixarum, quæ inde variabilis reddetur. Maximam vero mutationem ſubibunt eæ ſtellæ fixæ, quarum longitudo in nodos orbitæ Jovis incidit, & quæ eſt vel $9^s, 8^o$, vel $3^s, 8^o$; hæcque maxima mutatio ob inclinationem orbitæ Jovis $= 1^o, 19' 10''$ ſingulis annis valebit 0'', 16 = 9''', ſinguliſque ſeculis 16''; quæ propterea elapſo quovis ſeculo ita ſe habebit:

Si longitudo ſtellæ ſit $9^s, 8^o$

ejus latitudo $\left\{\begin{matrix}\text{decreſcit}\\ \text{creſcit}\end{matrix}\right\}$ 16'' ſi latitudo fuerit $\left\{\begin{matrix}\text{borealis}\\ \text{auſtralis}\end{matrix}\right\}$

At ſi longitudo ſtellæ ſit $3^s, 8^o$

ejus latitudo $\left\{\begin{matrix}\text{creſcit}\\ \text{decreſcit}\end{matrix}\right\}$ 16'' ſi latitudo fuerit $\left\{\begin{matrix}\text{borealis}\\ \text{auſtralis}\end{matrix}\right\}$

pro reliquis ſtellis fixis hæc mutatio ſecularis in latitudine diminui debet in ratione ſinus totius ad coſinum differentiæ longitudinis ſtellæ ab his duobus limitibus.